Published by Institute of Physics Publishing for SISSA
Received: *March 27, 2008*
Accepted: *June 10, 2008*
Published: *July 1, 2008*

JHEP07(2008)004

Mixed-symmetry massless fields in Minkowski space unfolded

E.D. Skvortsov

I.E.Tamm Department of Theoretical Physics, P.N.Lebedev Physical Institute, Leninsky prospect 53, 119991, Moscow, Russia
E-mail: skvortsov@lpi.ru

Abstract: The unfolded formulation for arbitrary massless mixed-symmetry bosonic and fermionic fields in Minkowski space is constructed. The unfolded form is proved to be uniquely determined by the requirement that all gauge symmetries are manifest. The unfolded equations have the form of a covariant constancy condition. The gauge fields and gauge parameters are differential forms with values in certain irreducible Lorentz tensors. The unfolded equations for bosons determine completely those for fermions. The proposed unfolded formulation also contains dual formulations for massless mixed-symmetry fields.

Keywords: Gauge Symmetry, Field Theories in Higher Dimensions.

Contents

JHEP07(2008)004

1. Introduction

In four dimensional Minkowski space spin degrees of freedom are known to be classified by non-negative integers or half-integers. However, in dimensions higher than four spin degrees of freedom are described by a set of (half)integers, according to the weights of the Wigner's little group. The simplest and the most developed are the cases of totally symmetric [1 – 11] and totally antisymmetric fields [12 – 14]. All other types are referred to collectively as mixed-symmetry. Mixed-symmetry fields naturally arise in field theories in higher-dimensions, for instance, in (super)string theory [15].

The simplest mixed-symmetry fields were originally considered in [16] and [17]. The most general type of mixed-symmetry fields was studied in [18 – 21] though, the rigorous proof of the fact that the proposed in [20] fields/gauge symmetries content and equations describe massless *particles* properly was given in [22, 23]. In terms of BRST approach mixed-symmetry fields, characterized by at most two non-zero weights, were studied in [24 – 27]. An elegant approach to the description of mixed-symmetry fields was proposed in [28 – 30] on the basis of the simplest mixed-symmetry fields.

In this paper massless mixed-symmetry fields are reformulated within the unfolded approach [31 – 33] because it is the unfolded approach that underlies the full nonlinear theory of interacting massless fields with arbitrary totally symmetric spins [7], being the only approach succeeded in constructing the full theory, though exhaustive results concerned with cubic vertices of higher-spin fields were obtained within the light-cone approach in [34 – 36]. Therefore, to unfold an arbitrary spin, viz., mixed-symmetry, fields in the Minkowski space is considered as the first step towards the full nonlinear theory of arbitrary spin fields.

The main statement of the paper is that a free massless field with spin degrees of freedom characterized by an arbitrary bosonic or fermionic unitary irreducible representation of the Wigner's little algebra can be uniquely described within the unfolded approach, in which all gauge symmetries are manifest. The unfolded system has the form of a covariant constancy equation. The gauge fields and gauge parameters are differential forms on the Minkowski space with values in certain irreducible representations of the Lorentz algebra, i.e., irreducible tensors or spin-tensors. The full unfolded system is described in terms of a single nilpotent operator σ_-, whose cohomology groups correspond to independent differential gauge parameters, dynamical fields, gauge-invariant equations and Bianchi identities.

Another advantage of the unfolded approach is in that the equations for bosons and fermions have literally the same form, the only difference being in change of tensor representations, in which the fields takes values, by the corresponding spin-tensors. The form and the order of dynamical equations, second for bosons and first for fermions, turns out to be encoded in σ_- cohomology. In fact the unfolded system is constructed for the bosonic case and, then, proved to have the same form for fermions. The similarity between bosons and fermions within the unfolded approach can have deep applications in theories with supersymmetries.

Despite the deep relations of the unfolded approach to the nonlinear theory of higher-spin fields, unfolding by itself provides a very powerfull method for analysis of dynamical systems. For instance, once some linear dynamical system is unfolded it is given a direct interpretation in terms of Lie algebras/modules and all gauge symmetries become manifest.

The paper is organized as follows: the main result, i.e., the unfolded form of equations describing a massless field with the spin that corresponds to an arbitrary irreducible representation of the Wigner's little algebra is stated in section 3. All the necessary information about mixed-symmetry fields in the Minkowski space-time is collected in section 4. The basic facts concerning the unfolded approach, viz., the very definition, the relation to Lie algebras/modules, to the Chevalley-Eilenberg cohomology are recalled in section 5, illustrated on the examples of a scalar field, spin-one field and totally symmetric spin-s and spin-$(s+\frac{1}{2})$ fields in section 5.3. The proof of the general statement of section 3 is in section 6. The physical degrees of freedom are analyzed in section 6.3. The discussion of the results and conclusions are in section 7. Multi-index notation and basic facts on Young diagrams and irreducible representations are collected in appendices.

JHEP07(2008)004

2. Conventions

As the most general type of irreducible representations of orthogonal algebras, viz., the Wigner's little algebra, the Lorentz algebra, is considered, the essential use is made of Young diagrams' language. A certain Young diagram is denoted by $\mathbf{Y}$ with subscripts or directly enumerating the lengths of the rows as $\mathbf{Y}\{s_1, s_2, \ldots\}$ or, when rows of equal lengths are combined to blocks, as $\mathbf{Y}\{(s_1, p_1), (s_2, p_2), \ldots\}$, p_i being the number of rows of length s_i. Loosely speaking we do not make any difference between irreducible finite-dimensional representations of orthogonal algebras, Young diagrams[1] and the corresponding irreducible (spin)-tensors,[2] e.g., rank-two symmetric traceless tensor-valued field ϕ^{ab}, i.e., $\phi^{ab} = \phi^{ba}$ and $\eta_{ab}\phi^{ab} = 0$, can be equivalently denoted either as $\phi^{\square\square}$ or $\phi^{\mathbf{Y}}$ with $\mathbf{Y} = \mathbf{Y}\{2\} \equiv \mathbf{Y}\{(2,1)\}$. The scalar representation $\mathbf{Y}\{0\}$ is denoted by $\bullet$. Unless otherwise stated, all Young diagrams are of orthogonal algebras, viz., $\mathfrak{so}(d-1,1)$ or $\mathfrak{so}(d-2)$. For more detail on Young diagrams see in appendix B. Greek indices μ, ν,...=0...(d-1) are the world indices of the Minkowski space-time $\mathcal{M}_d$. $d \equiv dx^\mu \frac{\partial}{\partial x^\mu}$ is the exterior differential on $\mathcal{M}_d$. The degree of differentials forms on $\mathcal{M}_d$ is indicated by the bold subscript, e.g., a degree-q differential form ω on $\mathcal{M}_d$ with values in $\mathfrak{so}(d-1,1)$-*irrep* characterized by the Young diagram $\mathbf{Y}$ is denoted as $\omega^{\mathbf{Y}}_{\mathbf{q}}$ (loosely speaking $\mathbf{Y}$-valued degree-q form $\omega^{\mathbf{Y}}_{\mathbf{q}}$). The wedge symbol $\wedge$ is systematically omitted. Lowercase Latin letters a, b,...=0...(d-1) are vector indices of $\mathfrak{so}(d-1,1)$, fiber indices of the sections of tensor bundles over the Minkowski space-time. Greek indices $\alpha, \beta, \gamma = 1 \ldots 2^{[\frac{d}{2}]}$ are fiber spinor indices of $\mathfrak{so}(d-1,1)$. The multi-index condensed notation is used in the paper: the (anti)-symmetrization is denoted by placing the corresponding indices in (square) round brackets, for details on the multi-index notation see appendix A.

3. Summary of results

The main statement of the paper is that given a unitary irreducible bosonic(fermionic) representation of the massless Wigner's little algebra $\mathfrak{so}(d-2)$, which is characterized by Young diagram $\mathbf{Y} = \mathbf{Y}\{(s_1, p_1), \ldots, (s_N, p_N)\}$ ($\mathbf{Y} = \mathbf{Y}\{(s_1, p_1), \ldots, (s_N, p_N)\}_{\frac{1}{2}}$), there exists a uniquely determined unfolded system that describes a massless spin-$\mathbf{Y}$ field, with all gauge symmetries being manifest. The system has the form of a covariant constancy equation

$$\mathcal{D}\omega_{\mathbf{p}} = 0, \qquad \omega_{\mathbf{p}} \in \mathcal{W}_{\mathbf{p}},$$
$$\delta\omega_{\mathbf{p}} = \mathcal{D}\xi_{\mathbf{p-1}}, \qquad \xi_{\mathbf{p-1}} \in \mathcal{W}_{\mathbf{p-1}},$$

[1] In addition to the Young symmetry conditions, extra restrictions (with the aid of invariant tensors: Levi-Civita for $sl(d)$, metric and Levi-Civita for $so(d)$) have to be imposed on the tensors to make them irreducible. In what follows it is important that irreducible $so(d)$-tensors are traceless. No special consideration is given to (anti)-self dual fields, see appendix B.

[2] In the case of fermionic representations of orthogonal algebras, i.e., spin-tensors, the tensor part of a spin-tensor (all but one spinor indices can be converted pairwise to tensor indices by means of Γ-matrices, hence, we consider spin-tensors with one spinor index only) is characterized by Young diagram, which is labeled by the subscript $\frac{1}{2}$, e.g., an irreducible rank-two symmetric tensor-spinor $\psi^{\alpha;ab}$ satisfies $\Gamma^{\alpha}_{a\,\beta}\psi^{\beta;ab} = 0$ and belongs to $\mathbf{Y}\{2\}_{\frac{1}{2}}$. The connection with the standard Gelfan-Zeitlin labels is obvious. Additional conditions, viz., Majorana, Weyl and Majorana-Weyl are irrelevant to the problems concerned.

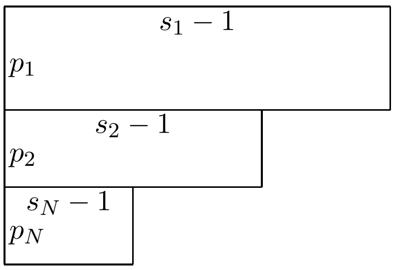
JHEP07(2008)004

$$\delta\xi_{\mathbf{p-1}} = \mathcal{D}\xi_{\mathbf{p-2}}, \qquad\qquad \xi_{\mathbf{p-2}} \in \mathcal{W}_{\mathbf{p-2}},$$
$$\ldots\ldots, \qquad\qquad \ldots\ldots,$$
$$\delta\xi_{\mathbf{1}} = \mathcal{D}\xi_{\mathbf{0}}, \qquad\qquad \xi_{\mathbf{0}} \in \mathcal{W}_{\mathbf{0}}, \tag{3.1}$$

where $\mathcal{W} = \bigoplus_{q=0}^{\infty} \mathcal{W}_{\mathbf{q}}$ is certain graded space, $\mathcal{D}$ is a nilpotent operator of degree $(+1)$, $\mathcal{D} : \mathcal{W}_{\mathbf{q}} \to \mathcal{W}_{\mathbf{q+1}}$ and $\mathcal{D}^2 = 0$. Gauge fields take values in $\mathcal{W}_{\mathbf{p}}$, where $p = \sum_{i=1}^{i=N} p_i$ is the height of Young diagram $\mathbf{Y}$, the first level gauge parameters in $\mathcal{W}_{\mathbf{p-1}}$, the second level gauge parameters in $\mathcal{W}_{\mathbf{p-2}}$ and so on. The gauge invariance is manifest by virtue of $\mathcal{D}^2 = 0$. The reducibility of gauge transformations is similar to those of totally anti-symmetric fields.

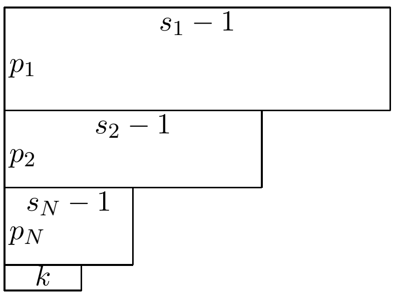
Space $\mathcal{W}_{\mathbf{p}}$, which contains the gauge fields of the unfolded system, is a graded by nonnegative integer $g = 0, 1, \ldots$ set of differential forms $\mathcal{W}_{\mathbf{p}} = \{\omega_{\mathbf{q_0}}^{\mathbf{Y}_0}, \omega_{\mathbf{q_1}}^{\mathbf{Y}_1}, \ldots, \omega_{\mathbf{q_g}}^{\mathbf{Y}_g}, \ldots\}$, $q_0 = p$. Diagrams $\mathbf{Y}_g$ that characterize $\mathfrak{so}(d-1,1)$-irreducible representations, in which the fields and gauge parameters take values, are uniquely determined by the initial diagram $\mathbf{Y} = \mathbf{Y}\{(s_1, p_1), \ldots, (s_N, p_N)\}_{(\frac{1}{2})}$ of $\mathfrak{so}(d-2)$.

The *dynamical* field is incorporated in a p-form $\omega_{\mathbf{p}}^{\mathbf{Y}_0} \in \mathcal{W}_{\mathbf{p}}^{g=0}$ that takes values in the irreducible representation(*irrep*) of the Lorentz algebra $\mathfrak{so}(d-1,1)$ that is characterized by Young diagram $\mathbf{Y}_0$ of the form



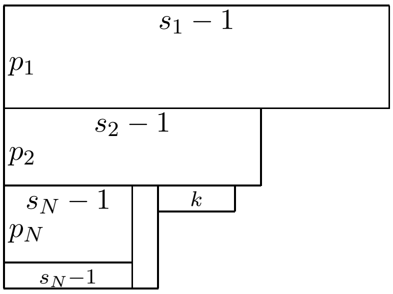
, (3.2)

i.e., it is obtained by cutting off the first column of $\mathbf{Y}$. All other gauge fields in the system are auxiliary and can be expressed in terms of derivatives of $\omega_{\mathbf{p}}^{\mathbf{Y}_0}$. It is convenient to enumerate the Lorentz-*irreps* in which gauge fields $\omega_{\mathbf{q_g}}^{\mathbf{Y}_g}$ take values by a pair $\{n, k\}$ of integers. Roughly speaking, the first integer is related to the number of the block of $\mathbf{Y}$, $n = N, \ldots, 0$, the second one is related to the relative length of the n-th and $(n+1)$-th blocks. The $\mathfrak{so}(d-1,1)$-*irreps* $\mathbf{Y}_{g=0}, \ldots, \mathbf{Y}_{g=(s_N-1)}$ are given by $\mathbf{Y}_{\{n,k\}}$ with $n = N$, $k = 0 \ldots (s_N - 1)$ of the form



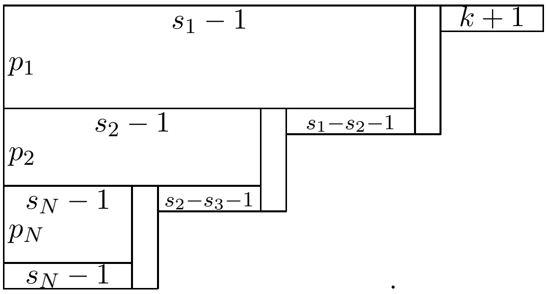
. (3.3)

The diagrams $\mathbf{Y}_g$ with $g = s_N \dots (s_{N-1} - 1)$ are given by $\mathbf{Y}_{\{n,k\}}$ with $n = N-1$, $k = 0 \dots (s_{N-1} - s_N - 1)$ of the form



. (3.4)

and analogously for the rest of $\mathbf{Y}_g$ with $g < s_1$. The diagrams $\mathbf{Y}_g$ with $g = s_1, s_1+1, \dots$ are given by $\mathbf{Y}_{\{n,k\}}$ with $n = 0$, $k = 0, 1, \dots$ of the form



. (3.5)

Gauge fields $\omega^{\mathbf{Y}_g}_{\mathbf{q_g}}$ for $g \sim \{n = N, k\}$ are $(p_1 + \dots + p_N)$-forms, gauge fields $\omega^{\mathbf{Y}_g}_{\mathbf{q_g}}$ for $g \sim \{n = N-1, k\}$ are $(p_1 + \dots + p_{N-1})$-forms, ..., gauge fields $\omega^{\mathbf{Y}_g}_{\mathbf{q_g}}$ for $g \sim \{n = 0, k\}$ are zero-forms. Hence, form degree function q_g is completely defined. The grade g is equal to the element number (starting from zero) in the set of pairs $\{n, k\}$ ordered by k in increasing order and, then, by n in decreasing order.

Space $\mathcal{W}_{\mathbf{p-1}}$, which contains the first level gauge parameters of the system, is a set of forms with values in the same $\mathfrak{so}(d-1,1)$-*irreps* as gauge fields but the form degree is less by one, i.e., $\mathcal{W}_{\mathbf{p-1}} = \{\xi^{\mathbf{Y}_0}_{\mathbf{q_0-1}}, \xi^{\mathbf{Y}_1}_{\mathbf{q_1-1}}, \dots, \xi^{\mathbf{Y}_g}_{\mathbf{q_g-1}}, \dots\}$. The sector $\mathcal{W}^g_{\mathbf{p-1}}$ for $g \geq s_1$ is trivial du to $q_g = 0$. Analogously, spaces $\mathcal{W}_{\mathbf{q}}$ for $q < p-1$ and $q > p$ can be defined. To sum up, the element of space $\mathcal{W}_{\mathbf{p \pm i}}$ at grade $g \sim \{n, k\}$ is a degree-$(q_g \pm i)$ form with values in $\mathbf{Y}_g \equiv \mathbf{Y}_{\{n,k\}}$ *irrep* of the Lorentz algebra.

The Minkowski background space is described in terms of vielbein(tetrad) $h^a_\mu dx^\mu$ and Lorentz spin-connection $\varpi^{a,b}_\mu dx^\mu$, which determines Lorentz-covariant derivative $D_L = d + \varpi$.

When reduced to $\mathcal{W}^g_{\mathbf{q}}$ the full system has the form

$$\begin{aligned}
D_L \omega^{\mathbf{Y}_g}_{\mathbf{q_g}} &= \sigma_- \left(\omega^{\mathbf{Y}_{g+1}}_{\mathbf{q_{g+1}}}\right), & \omega^{\mathbf{Y}_g}_{\mathbf{q_g}} &\in \mathcal{W}^g_{\mathbf{p}}, & \omega^{\mathbf{Y}_{g+1}}_{\mathbf{q_{g+1}}} &\in \mathcal{W}^{g+1}_{\mathbf{p}}, \\
\delta \omega^{\mathbf{Y}_g}_{\mathbf{q_g}} &= D_L \xi^{\mathbf{Y}_g}_{\mathbf{q_g-1}} + \sigma_- \left(\xi^{\mathbf{Y}_{g+1}}_{\mathbf{q_{g+1}-1}}\right), & \xi^{\mathbf{Y}_g}_{\mathbf{q_g-1}} &\in \mathcal{W}^g_{\mathbf{p-1}}, & \xi^{\mathbf{Y}_{g+1}}_{\mathbf{q_{g+1}-1}} &\in \mathcal{W}^{g+1}_{\mathbf{p-1}}, \\
\delta \xi^{\mathbf{Y}_g}_{\mathbf{q_g-1}} &= \cdots ,
\end{aligned} \tag{3.6}$$

where operator $\mathcal{D}$ is a sum $\mathcal{D} = D_L - \sigma_-$ of Lorentz-covariant derivative D_L and certain nilpotent operator $\sigma_- : \bigwedge^q \otimes \mathbf{Y}_{g+1} \to \bigwedge^{q+\Delta_g+1} \otimes \mathbf{Y}_g$, $\Delta_g = q_g - q_{g+1} \geq 0$, $(\sigma_-)^2 = 0$, built of background vielbein $h^a_\mu dx^\mu$, which is unambiguously fixed by the symmetry of $\mathbf{Y}_g$ and $\mathbf{Y}_{g+1}$. σ_- contracts $(\Delta_g + 1)$ vielbeins h^a with the tensor representing $\mathbf{Y}_{g+1}$ to obtain the tensor with the symmetry of $\mathbf{Y}_g$, appropriate Young symmetrizers are implied.

Since σ_- affects tensor indices only it is correctly defined on spin-tensors too and does not violate the Γ-tracelessness condition. The unfolded equations for fermions have the same form as for bosons, the irreducible tensors are to be replaced with corresponding spin-tensors.

The case of the last block of the length one, i.e., single column, is not special but requires some comments. Since $(s_N - 1) = 0$, it is not possible to add a cell to the bottom-left of the N-th block of $\mathbf{Y}_0$, therefore, $\mathbf{Y}_{\{n=N,k=0\}} \equiv \mathbf{Y}_0$ is the only diagram with $n = N$ and diagram $\mathbf{Y}_{\{n=N-1,k=0\}} \equiv \mathbf{Y}_{g=1}$ has the symmetry of $\mathbf{Y}\{(s_1-1,p_1),\dots,(s_{N-1}-1,p_{N-1}),(1,p_N+1)\}$.

Subspace $\mathcal{W}^{\{n,k\}}_{\mathbf{q}}$ with definite n forms an irreducible module of $\mathfrak{iso}(d-1,1)$, whereas the subspace with definite both n and k forms a finite-dimensional irreducible module of $\mathfrak{so}(d-1,1)$, i.e., an irreducible Lorentz tensor, characterized by $\mathbf{Y}_g$, as was stated above. The intervals of constancy of q_g correspond to $g_{n,k}$ with definite n, i.e., the set of forms, on which a certain irreducible $\mathfrak{iso}(d-1,1)$-module is realized, all have the same degree.

Let us also note that $\mathcal{W}_{\mathbf{k}\geq\mathbf{p}}$ are infinite-dimensional, whereas $\mathcal{W}_{\mathbf{k}<\mathbf{p}}$ are finite-dimensional. The higher degree spaces $\mathcal{W}_{\mathbf{k}\geq\mathbf{p}}$ correspond to the equations of motion ($\mathcal{W}_{\mathbf{k}=\mathbf{p}+\mathbf{1}}$) and Bianchi identities ($\mathcal{W}_{\mathbf{k}>\mathbf{p}+\mathbf{1}}$), which manifest the gauge symmetries. Most of the equations express higher grade $g > 0$ fields via the derivatives of *physical* field $\omega^{\mathbf{Y}_0}_{\mathbf{q_0}}$ and only certain elements of $\mathcal{W}^{g=2}_{\mathbf{k}=\mathbf{p}+\mathbf{1}}$ impose on $\omega^{\mathbf{Y}_0}_{\mathbf{q_0}}$ second order dynamical equations. The significance of the fields with $g > 0$ is to make all gauge symmetries be manifest.

4. Mixed-symmetry fields in Minkowski space

The types of the Minkowski space *particles*, being, by definition, in one-to-one correspondence with unitary irreducible representations(*uirrep*) of the Poincare algebra $\mathfrak{iso}(d-1,1)$, in the case of four space-time dimensions were classified by Wigner in [37].

Leaving out the details of the Wigner's construction, for recent reviews and for generalization to an arbitrary space-time dimension d see [38], important is that given $m^2 \geqslant 0$ and a *uirrep* $\mathbf{Y}$ of the Wigner's little algebra, being $\mathfrak{so}(d-1)$ for $m^2 > 0$ and $\mathfrak{so}(d-2)$ for $m^2 = 0$, there exists a standard procedure to construct a *uirrep* of $\mathfrak{iso}(d-1,1)$, which is called a massive(massless) *particle* of spin-$\mathbf{Y}$. So-called continuous or infinite spin particles [37, 39] are not considered in this paper. Therefore, physical degrees of freedom for massive and massless *particles* are classified by irreducible tensors of $\mathfrak{so}(d-1)$ and $\mathfrak{so}(d-2)$, respectively.

More elaborated are the two cases of totally-symmetric spin-s *particles* $\mathbf{Y} = \mathbf{Y}\{s\} \equiv \mathbf{Y}\{(s,1)\}$ [40, 1, 10], viz., scalar, vector, graviton, and of totally anti-symmetric spin-p *particles* $\mathbf{Y} = \mathbf{Y}\{(1,p)\}$ [12–14]. The others are referred to collectively as mixed-symmetry, the simplest one being $\mathbf{Y} = \mathbf{Y}\{2,1\} \equiv \mathbf{Y}\{(2,1),(1,1)\}$.

Yet different problem is to realize a *uirrep* of $\mathfrak{iso}(d-1,1)$ on the solutions of a wave equation for a field $\phi_{\mathbf{Y}_M}(x)$, which takes values in a certain representation $\mathbf{Y}_M$ of the Lorentz algebra $\mathfrak{so}(d-1,1)$, i.e., $\phi_{\mathbf{Y}_M}(x)$ is a Lorentz tensor or a set of tensors. As field theories free *particles* can be described in either non-gauge or gauge way, in the former case a *uirrep* of $\mathfrak{iso}(d-1,1)$ is realized on the solutions of the wave equation directly, whereas for the latter case a *uirrep* of $\mathfrak{iso}(d-1,1)$ is realized on the quotient of the solutions by certain specific solutions, called pure gauge. The wave equation $(\Box+m^2)\phi_{\mathbf{Y}_M}(x)=0$, which fixes the quadratic Casimir m^2 of $\mathfrak{iso}(d-1,1)$, is generally supplemented with a set of algebraic/differential constraints to single out an irreducible, in the sense of the little algebra, component. Field $\phi_{\mathbf{Y}_M}(x)$ takes values in a certain finite-dimensional representation of $\mathfrak{so}(d-1,1)$, which is not irreducible in most cases, nevertheless it can be made irreducible when dealing with free equations of motion only. Yet more different problem is to realize an *irrep* of $\mathfrak{iso}(d-1,1)$ on the solutions of the variational problem for some action, in most cases the procedure requires a set of auxiliary fields, which carry no physical degrees of freedom.

The choice of $\mathfrak{so}(d-1,1)$-representation $\mathbf{Y}_M$ (even if irreducible), in which field $\phi_{\mathbf{Y}_M}(x)$ takes values, is not unique, e.g., a free massless spin-one *particle* can be described either by a gauge potential $A_\mu(x)$ subjected to

$$\Box A_\mu-\partial_\mu\partial^\nu A_\nu=0,\qquad \delta A_\mu=\partial_\mu\xi, \tag{4.1}$$

or by a field strength $F_{\mu\nu}$ subjected to

$$\partial^\mu F_{\mu\nu}=0,\quad \partial_{[\mu}F_{\lambda\rho]}=0. \tag{4.2}$$

In the case of a massless spin-two *particle* ($\mathbf{Y}=\Box\!\Box_{\mathfrak{so}(d-2)}$), in addition to the conventional description in terms of the metric $g_{\mu\nu}$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=0 \tag{4.3}$$

the Weyl tensor $C_{\mu\nu,\lambda\rho}$ known to have the symmetry of $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}$[3] can describe a free spin-two in a non gauge way

$$\partial_{[\mu}C_{\mu\mu],\nu\nu}=0,\quad \partial^\lambda C_{\mu\lambda,\nu\nu}=0. \tag{4.4}$$

Another example is a $4d$ massless scalar particle, which can be described either by a scalar field $\phi(x)$ subjected to $\Box\phi=0$ or, more exotically, by an antisymmetric gauge field $\omega_{\mu\nu}$ (so-called notoph [41]) subjected to

$$\Box\omega_{\mu\nu}-\partial_\mu\partial^\rho\omega_{\rho\nu}+\partial_\nu\partial^\rho\omega_{\rho\mu}=0,\quad \delta\omega_{\mu\nu}=\partial_\mu\xi_\nu-\partial_\nu\xi_\mu. \tag{4.5}$$

This equation describes a massless *particle* with spin-$\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}_{\mathfrak{so}(d-2)}\equiv\mathbf{Y}\{1,1\}_{\mathfrak{so}(d-2)}$, which for $d=4$ by virtue of the $\mathfrak{so}(d-2)$ Levi-Civita tensor ε_{ij} is equivalent to a scalar, $\mathbf{Y}\{1,1\}_{\mathfrak{so}(d-2)}\sim\mathbf{Y}\{0,0\}_{\mathfrak{so}(d-2)}$. On the other hand, a scalar *particle* can be described by

[3]In antisymmetric basis $C_{\mu\nu,\lambda\rho}$ satisfies $C_{\mu\nu,\lambda\rho}=-C_{\nu\mu,\lambda\rho}$, $C_{\mu\nu,\lambda\rho}=-C_{\mu\nu,\rho\lambda}$ and $C_{[\mu\nu,\lambda]\rho}=0$.

JHEP07(2008)004

a rank-d antisymmetric field $\omega_{\mu_1 \ldots \mu_d}$ satisfying $\Box\omega_{\mu_1\ldots\mu_d} = 0$, where the use of $\mathfrak{so}(d-1,1)$-duality is made of, $\omega_{\mu_1\ldots\mu_d} = \varepsilon_{\mu_1\ldots\mu_d}\phi$. These two types of duality are referred to as trivial.

The general statement is that free *particles* can be described by an infinite number of ways, called *dual* descriptions [42–46], but *dual* theories exhibit certain difficulties while introducing interactions [47–50], e.g., despite the fact that free massless spin-one and spin-two *particles* can be described by the Maxwell field strength and by the Weyl tensor, respectively, introducing interactions requires the corresponding gauge potentials A_μ and $g_{\mu\nu}$ to be brought in.

There exists a distinguished choice of $\mathfrak{so}(d-1,1)$-*irrep* $\mathbf{Y}_M$, in which field $\phi_{\mathbf{Y}_M}(x)$ takes values, that can be referred to as fundamental or minimal. For the minimal description of a spin-$\mathbf{Y}$ *particle* the spin degrees of freedom and the $\mathfrak{so}(d-1,1)$-*irrep* $\mathbf{Y}_M$ are characterized by the same Young diagram, i.e., $\mathbf{Y}_M = \mathbf{Y}$, e.g., a spin-one particle by A_μ, a spin-two by $g_{\mu\nu}$. All other descriptions are referred to as *dual*. It is the minimal descriptions that will be discussed further, by this reason the term spin-$\mathbf{Y}$ *particle* can be substituted for more accepted spin-$\mathbf{Y}$ *field*.

A massive totally symmetric spin-s *field* can be described [51] by a totally symmetric tensor field $\phi_{(\mu_1\ldots\mu_s)}$ subjected to

$$\begin{aligned}
(\Box + m^2)\phi_{\mu_1\ldots\mu_s} &= 0,\\
\partial^\nu \phi_{\nu\mu_2\ldots\mu_s} &= 0,\\
\phi^\nu{}_{\nu\mu_3\ldots\mu_s} &= 0,
\end{aligned} \tag{4.6}$$

where the last equation (tracelessness condition) makes the tensor irreducible in the $\mathfrak{so}(d-1,1)$ sense, the first one puts the system *on-mass-shell* and the second one projects out the components orthogonal to the momentum, restricting $\phi_{\mu_1\ldots\mu_s}$ to contain only a spin-s *irrep* of $\mathfrak{so}(d-1)$.

Analogously, a massive totally anti-symmetric spin-p *field*, i.e., $\mathbf{Y} = \mathbf{Y}\{(1,p)\}$, can be described by a totally anti-symmetric tensor field $\omega_{[\mu_1\ldots\mu_p]}$ subjected to

$$\begin{aligned}
(\Box + m^2)\omega_{\mu_1\ldots\mu_p} &= 0,\\
\partial^\nu \omega_{\nu\mu_2\ldots\mu_p} &= 0,
\end{aligned} \tag{4.7}$$

where the tracelessness condition becomes trivial for anti-symmetric tensors.

These two results are easily generalized to an arbitrary spin-$\mathbf{Y}\{s_1,\ldots,s_n\}$ massive *field*, which can be minimally described by a symmetric in each group of indices tensor field $\phi_{\mu_1(s_1),\mu_2(s_2),\ldots,\mu_n(s_n)}$ subjected to

$$\begin{aligned}
(\Box + m^2)\phi_{\mu_1(s_1),\mu_2(s_2),\ldots,\mu_n(s_n)} &= 0,\\
\partial^{\mu_i}\phi_{\mu_1(s_1),\mu_2(s_2),\ldots,\mu_n(s_n)} &= 0, \qquad i\in[1,n],\\
\phi_{\mu_1(s_1),\ldots,(\mu_k(s_k),\ldots,\mu_k)\mu_i(s_i-1),\ldots,\mu_n(s_n)} &= 0, \qquad k\in[1,n-1], \quad k<i,\\
\eta^{\mu_i\mu_j}\phi_{\mu_1(s_1),\mu_2(s_2),\ldots,\mu_n(s_n)} &= 0, \quad i,j\in[1,n],
\end{aligned} \tag{4.8}$$

where the last two conditions are just the Young symmetry and the tracelessness conditions, which make the field carry an *irrep*-$\mathbf{Y}\{s_1,\ldots,s_n\}$ of $\mathfrak{so}(d-1,1)$, and can be thought of as the part of the definition of field $\phi_{\mathbf{Y}_M}(x)$. The first equation puts the system *on-mass-shell*, the second one projects out all $\mathfrak{so}(d-1)$-*irreps*, which the tensor $\phi_{\mu_1(s_1),\mu_2(s_2),\dots,\mu_p(s_p)}$ decomposes into, except for the one with the symmetry of $\mathbf{Y}\{s_1,\dots,s_n\}$ $\mathfrak{so}(d-1)$-diagram.

Let $\mathcal{D}\left(m^2;\mathbf{Y}\{s_1,\dots,s_n\}\right)$ be an $\mathfrak{iso}(d-1,1)$ module extracted by (4.8), being irreducible for $m^2\neq 0$ or for $\mathbf{Y}\{0\}$ (scalar *field*). Since minimally described massless *fields* are gauge theories a *uirrep* $\mathcal{H}\left(0;\mathbf{Y}\{s_1,\dots,s_n\}\right)$ of $\mathfrak{iso}(d-1,1)$ corresponding to a massless spin-$\mathbf{Y}\{s_1,\dots,s_n\}$ *field* should be defined as appropriate quotient by the pure gauge solutions of the form

$$0\longrightarrow \text{gauge solutions}\longrightarrow \text{all solutions}\longrightarrow \mathcal{H}\left(0;\mathbf{Y}\{s_1,\dots,s_n\}\right)\longrightarrow 0, \tag{4.9}$$

where the sequence is non-split, as there is no $\mathfrak{so}(d-1,1)$-covariant way to extract $\mathfrak{so}(d-2)$-*irrep* $\mathbf{Y}\{s_1,\dots,s_n\}$ from the Lorentz tensor with the same symmetry of $\mathbf{Y}\{s_1,\dots,s_n\}$ by virtue of a single $\mathfrak{so}(d-1,1)$-vector, the momentum $p_\mu\sim\partial_\mu$.

A massless totally symmetric spin-s *field* can be described as the quotient of (4.6) with $m^2=0$ by pure gauge solutions of the form $\delta\phi_{\mu_1\dots\mu_s}=\partial_{(\mu_1}\xi_{\mu_2\dots\mu_s)}$, where $\xi_{\mu_1\dots\mu_{s-1}}$ is a totally symmetric gauge parameter subjected to the equations of the same form (4.6), i.e., *on-mass-shell*, tracelessness and transversality, thus, belonging to $\mathcal{D}\left(0;\mathbf{Y}\{s-1\}\right)$. The definition of $\mathcal{H}\left(0;\mathbf{Y}\{s\}\right)$ is given by

$$0\longrightarrow \mathcal{D}\left(0;\mathbf{Y}\{(s-1,1)\}\right)\longrightarrow \mathcal{D}\left(0;\mathbf{Y}\{(s,1)\}\right)\longrightarrow \mathcal{H}\left(0;\mathbf{Y}\{(s,1)\}\right)\longrightarrow 0. \tag{4.10}$$

There is a bit difference for a totally anti-symmetric spin-p massless *field*. Pure gauge solutions are defined analogously as $\delta\omega_{\mu_1\dots\mu_p}=\partial_{[\mu_1}\xi_{\mu_2\dots\mu_p]}$, where $\xi_{[\mu_1\dots\mu_{p-1}]}$ is a rank-$(p-1)$ antisymmetric gauge parameter subjected to the equation of the same form (4.7) and, thus, belonging to $\mathcal{D}\left(0;\mathbf{Y}\{(1,p-1)\}\right)$. In contrast to a totally symmetric spin-s massless *field*, the gauge transformations are reducible in the sense that $\delta\omega_{\mu_1\dots\mu_p}\equiv 0$ provided that one transforms gauge parameter $\delta\xi_{\mu_1\dots\mu_{p-1}}=\partial_{[\mu_1}\xi_{\mu_2\dots\mu_{p-1}]}$ with a second level rank-$(p-2)$ antisymmetric gauge parameter $\xi_{[\mu_1\dots\mu_{p-2}]}$, and so on. Some components (parallel to the momentum) of the first level gauge parameter $\xi_{\mu_1\dots\mu_{p-1}}$ do not contribute to the gauge law for $\omega_{\mu_1\dots\mu_p}$, these are represented by the second level gauge parameter $\xi_{\mu_1\dots\mu_{p-2}}$ modulo those components of $\xi_{\mu_1\dots\mu_{p-2}}$ that do not contribute to $\xi_{\mu_1\dots\mu_{p-1}}$ and so on till $\delta\xi_\mu=\partial_\mu\xi$. Gauge parameters $\xi_{\mu_1\dots\mu_{p-1}}$, $\xi_{\mu_1\dots\mu_{p-k}}$, $k\in[1,p]$ and ξ are referred to as the first level, the k-th level and the deepest level of reducibility, respectively. The corresponding $\mathfrak{iso}(d-1,1)$-*uirrep* is given by a non-split exact sequence of the form

$$\begin{aligned}0\longrightarrow \mathcal{D}\left(0;\mathbf{Y}\{(0,0)\}\right)&\longrightarrow \mathcal{D}\left(0;\mathbf{Y}\{(1,1)\}\right)\longrightarrow\\ \dots\dots&\longrightarrow \mathcal{D}\left(0;\mathbf{Y}\{(1,p)\}\right)\longrightarrow \mathcal{H}\left(0;\mathbf{Y}\{(1,p)\}\right)\longrightarrow 0.\end{aligned} \tag{4.11}$$

For example, Maxwell gauge potential A_μ possesses gauge parameters of the first level only as $p=1$ in this case.

Though, a considerable success in describing *on-mass-shell* massless *fields* was achieved, for instance, within the light-cone approach [52], there are many reasons to have an *off-shell* gauge symmetry, i.e., to construct equations that are invariant with respect to gauge transformations with gauge parameters not subjected to $\Box\xi = 0$. To make the symmetry *off-shell* generally requires to relax the irreducibility of the $\mathfrak{so}(d-1,1)$-representation in which field $\phi_{\mathbf{Y}}(x)$ takes values.

For instance, a massless totally symmetric spin-s field can be described [53] by a traceless rank-s symmetric field $\phi_{\mu_1\ldots\mu_s}$ with an *off-shell* gauge symmetry

$$\Box\phi_{\mu_1\ldots\mu_s} - s\partial_{(\mu_1}\partial^{\nu}\phi_{\nu\mu_2\ldots\mu_s)} + \tfrac{s(s-1)}{(d+2s-4)}\eta_{(\mu_1\mu_2}\partial^{\nu}\partial^{\rho}\phi_{\nu\rho\mu_3\ldots\mu_s)} = 0, \qquad \phi^{\nu}{}_{\nu\mu_3\ldots\mu_s} = 0,$$
$$\delta\phi_{\mu_1\ldots\mu_s} = \partial_{(\mu_1}\xi_{\mu_2\ldots\mu_s)}, \qquad \xi^{\nu}{}_{\nu\mu_3\ldots\mu_{s-1}} = 0, \qquad \partial^{\nu}\xi_{\nu\mu_2\ldots\mu_{s-1})} = 0, \quad (4.12)$$

where in order for $\phi_{\mu_1\ldots\mu_s}$ to be traceless the gauge parameter has to be not only traceless but transverse also, this still being true for general mixed-symmetry fields. Apparently, to get rid of any differential constraints on gauge parameters, the tracelessness constraint for field $\phi_{\mu_1\ldots\mu_s}$ has to be relaxed. The same spin-s massless *field* can be described [1] by field $\phi_{\mu_1\ldots\mu_s}$ subjected to[4]

$$\Box\phi_{\mu_1\ldots\mu_s} - s\partial_{(\mu_1}\partial^{\nu}\phi_{\nu\mu_2\ldots\mu_s)} + \tfrac{s(s-1)}{2}\partial_{(\mu_1}\partial_{\mu_2}\phi^{\nu}{}_{\nu\mu_3\ldots\mu_s)} = 0, \qquad \phi^{\nu\rho}{}_{\nu\rho\mu_5\ldots\mu_s} = 0,$$
$$\delta\phi_{\mu_1\ldots\mu_s} = \partial_{(\mu_1}\xi_{\mu_2\ldots\mu_s)}, \quad \xi^{\nu}{}_{\nu\mu_3\ldots\mu_{s-1}} = 0, \quad (4.13)$$

where in order to get rid of any differential constraints on gauge parameters a tracelessness is relaxed to a double-tracelessness, i.e., field $\phi_{\mu_1\ldots\mu_s}$ takes values in the direct sum of two $\mathfrak{so}(d-1,1)$-*irreps*, being symmetric traceless tensors of ranks s and $(s-2)$. The general statement is that for equations of motion to have an *off-shell* gauge symmetry the $\mathfrak{so}(d-1,1)$-*irrep* in which field $\phi_{\mathbf{Y}}(x)$ takes values has to be reducible, with additional direct summands representing certain nonzero traces. These additional fields are called auxiliary and carry no physical degrees of freedom. Imposing certain gauge, equations (4.8) can be restored.

A massless totally antisymmetric spin-p *field* can be described by an antisymmetric rank-p tensor field $\omega_{[\mu_1\ldots\mu_p]}$ subjected to

$$\Box\omega_{\mu_1\ldots\mu_p} - p\partial_{[\mu_1}\partial^{\nu}\omega_{\nu\mu_2\ldots\mu_p]} = 0, \quad \delta\omega_{\mu_1\ldots\mu_p} = \partial_{[\mu_1}\xi_{\mu_2\ldots\mu_p]}, \quad (4.14)$$

where the symmetry at the all levels of reducibility is manifest and *off-shell*, i.e., $\delta\xi_{\mu_1\ldots\mu_{p-1}} = \partial_{[\mu_1}\xi_{\mu_2\ldots\mu_{p-1}]}$ for the second level gauge parameter $\xi_{\mu_1\ldots\mu_{p-2}}$ not subjected to any differential constraints, and analogously for the gauge symmetries at deeper levels.

Much similar to totally anti-symmetric fields mixed-symmetry massless fields possess reducible gauge transformations, i.e., for a field ϕ with equations of motion invariant under gauge transformations $\delta_{\xi_1}\phi = \sum_{i_1}\partial\xi_1^{i_1}$ there exist the second level gauge transformations $\delta_{\xi_2}\xi_1^{i_1} = \sum_{i_2}\partial\xi_2^{i_2}$ such that $\delta_{\xi_2}\phi \equiv 0$, there exist the third level gauge transformations $\delta_{\xi_3}\xi_2^{i_2} = \sum_{i_3}\partial\xi_3^{i_3}$ such that $\delta_{\xi_3}\xi_1 \equiv 0$ and so on. The difference from totally anti-symmetric fields is in that there are generally more than one gauge parameters at each level of reducibility (enumerated by index i_k at the k-th level).

[4] Obtained from the Lagrangian, equations of [1] have the form $G_{\mu_1\ldots\mu_s} - \frac{s(s-1)}{4}\eta_{(\mu_1\mu_2}G^{\nu}{}_{\nu\mu_3\ldots\mu_s)} = 0$, where $G_{\mu_1\ldots\mu_s}$ is equal to (4.13), the two forms being equivalent.

For instance, the simplest mixed-symmetry massless *field* has the spin-$\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}$ and can be minimally described [16, 17] by a field $\phi_{[\mu\mu],\nu}$, which is anti-symmetric in the first two indices and satisfies Young symmetry condition[5] $\phi_{[\mu\mu,\nu]}=0$. The equations of motion

$$\Box\phi_{\mu\mu,\lambda}+2\partial_{[\mu}\partial^{\lambda}\phi_{\mu]\lambda,\nu}-\partial_{\nu}\partial^{\lambda}\phi_{\mu\mu,\lambda}-2\partial_{\nu}\partial_{[\mu}\phi_{\mu]\lambda,}{}^{\lambda}=0 \tag{4.15}$$

are invariant under

$$\delta\phi_{\mu\mu,\nu}=\partial_{[\mu}\xi^{S}_{\mu]\nu}+\partial_{[\mu}\xi^{A}_{\mu]\nu}-\partial_{\nu}\xi^{A}_{\mu\mu}, \tag{4.16}$$

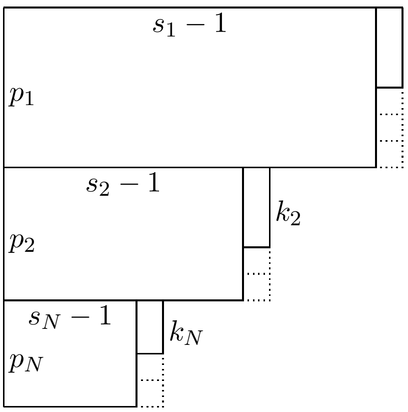
with symmetric and anti-symmetric gauge parameters $\xi^S_{(\mu\nu)}$ and $\xi^A_{[\mu\nu]}$. Let us stress that to maintain an *off-shell* gauge symmetry field $\phi_{[\mu\mu],\nu}$ has to take values in a reducible $\mathfrak{so}(d-1,1)$ representation, $\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}\oplus\square$, so does symmetric gauge parameter $\xi^S_{(\mu\mu)}$, $\square\!\square\oplus\bullet$, which is in accordance with $\phi_{\mu\nu,}{}^{\nu}\neq0$ and $\xi^{S\nu}_{\nu}\neq0$. Analogously to totally anti-symmetric fields there exist second level gauge transformations with a vector parameter ξ_μ

$$\begin{aligned}\delta\phi_{\mu\mu,\nu}&=0,\\ \delta\xi^{A}_{\mu\nu}&=\tfrac{2}{3}(\partial_\mu\xi_\nu-\partial_\nu\xi_\mu),\\ \delta\xi^{S}_{\mu\nu}&=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu.\end{aligned} \tag{4.17}$$

In this case $\mathcal{H}\left(0;\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}\right)$ is given by a non-split exact sequence of the form

$$0\longrightarrow\mathcal{D}\left(0;\square\right)\longrightarrow\mathcal{D}\left(0;\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\oplus\mathcal{D}\left(0;\square\!\square\right)\longrightarrow\mathcal{D}\left(0;\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}\right)\longrightarrow\mathcal{H}\left(0;\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}\right)\longrightarrow0 \tag{4.18}$$

In the general case of a massless spin-$\mathbf{Y}=\mathbf{Y}\{(s_1,p_1),\dots,(s_N,p_N)\}$ *field* the depth of reducibility of gauge transformations is equal to $p=\sum_{i=1}^{i=N}p_i$, where p is the height of the first column of the Young diagram and at the r-th level of reducibility gauge parameters have the symmetry of



$$:\quad\sum_{i=1}^{i=N}k_i=\sum_{i=1}^{i=N}p_i-r \tag{4.19}$$

This pattern of reducibility of gauge transformations will be of great importance while constructing the unfolded formulation in section 6. As is easily seen, at the first level of

[5] In more detail, the field $\phi_{\mu\nu,\lambda}$ satisfies $\phi_{\mu\nu,\lambda}=-\phi_{\nu\mu,\lambda}$, $\phi_{\mu\nu,\lambda}+\phi_{\nu\lambda,\mu}+\phi_{\lambda\mu,\nu}=0$. Equivalently, a symmetric basis can be used, i.e, $\phi^S_{\mu\nu,\lambda}=\phi^S_{\nu\mu,\lambda}$, $\phi^S_{\mu\nu,\lambda}+\phi^S_{\nu\lambda,\mu}+\phi^S_{\lambda\mu,\nu}=0$. The two bases are related by $\phi^S_{(\mu\nu),\lambda}=\frac{1}{\sqrt{3}}(\phi^A_{\mu\lambda,\nu}+\phi^A_{\nu\lambda,\mu})$, $\phi^A_{[\mu\nu],\lambda}=\frac{1}{\sqrt{3}}(\phi^S_{\mu\lambda,\nu}-\phi^S_{\nu\lambda,\mu})$.

reducibility, the gauge parameters are various tensors, whose Lorentz Young diagrams are obtained by cutting off one cell from the original $\mathfrak{so}(d-2)$-diagram in all possible ways, i.e., the number of gauge parameters at the first level is equal to the number of blocks N, e.g., two for spin-$\yng(2,1)$. There is only one gauge parameter at the deepest level of reducibility, whose Young diagram is obtained by cutting off the first column from $\mathbf{Y}$, e.g., $\square$ for spin-$\yng(2,1)$.

This pattern corresponds, of course, to *on-shell* equations, i.e., the gauge parameters taking values in $\mathfrak{so}(d-1,1)$-*irreps* of Young symmetry (4.19) are subjected to (4.8)-like equations. To obtain an *off-shell* gauge symmetry the field content has to be extended to take values in certain reducible $\mathfrak{so}(d-1,1)$-representations, additional components turn out can be identified with certain traces of a single field with the symmetry of $\mathbf{Y}$ as an $sl(d)$-tensor. In general, gauge parameters are also reducible tensors with the symmetry of (4.19). In the case of a spin-$\yng(2,1)$ massless *field*, the trace $\phi_{\mu\nu,}{}^{\nu}$ in the sector of fields and the trace $\xi^{S\nu}_{\nu}$ in the sector of gauge parameters are the additional components. In the general case of a spin-$\mathbf{Y}=\mathbf{Y}\{s_1,\ldots,s_n\}$ massless *field*, field $\phi_{\mu_1(s_1),\mu_2(s_2),\ldots,\mu_n(s_n)}$ should satisfy [20]

$$\eta^{\mu_i\mu_i}\eta^{\mu_i\mu_i}\phi_{\mu_1(s_1),\mu_2(s_2),\ldots,\mu_n(s_n)}=0,\qquad i\in[1,n], \tag{4.20}$$

which is a generalization of the Fronsdal's double-trace condition (4.13). As it will be shown, this condition naturally arises in the unfolded approach. Note that double-tracelessness is imposed on each group of symmetric indices and it is not required for cross-traces to vanish.

Let a generalized Weyl tensor for a minimally described spin-$\mathbf{Y}$ massless *field* be a gauge-invariant combination of the least order in derivatives of field $\phi_{\mathbf{Y}}(x)$ that is allowed to be nonzero *on-mass-shell.* On the other hand it is the generalized Weyl tensor that the minimal non-gauge description of a massless spin-$\mathbf{Y}$ *field* is based on. For a spin-$\mathbf{Y}=\mathbf{Y}\{(s_1,p_1),\ldots,(s_N,p_N)\}_{\mathfrak{so}(d-2)}$ massless *field* the generalized Weyl tensor has the symmetry of $\mathbf{Y}\{(s_1,p_1+1),(s_2,p_2),\ldots,(s_N,p_N)\}_{\mathfrak{so}(d-1,1)}$ and is of the s_1-th order in derivatives. In the case of spin-one ($\mathbf{Y}=\square_{\mathfrak{so}(d-2)}$) Maxwell field strength $F_{\mu\nu}$ with the symmetry of $\yng(1,1)$ can be also called a generalized Weyl tensor.

Fermionic mixed-symmetry fields share most features of bosonic ones, viz., the reducibility of gauge transformations, the enlargement of the field content for the equations of motion to possess an *off-shell* gauge invariance. The difference is that the equations for fermions have the first order in derivatives and the irreducibility of spin-tensors is achieved by the Γ-tracelessness condition[6] instead of the tracelessness one.

For example, a massless totally symmetric spin-$(s+\frac{1}{2})$ field can be described *off-shell* [54] by a totally symmetric spin-tensor field $\phi_{\alpha;(\mu_1\ldots\mu_s)}$ subjected to[7]

$$\begin{aligned}
\not\partial\phi_{\mu_1\ldots\mu_s}-s\partial_{(\mu_1}\Gamma^{\nu}\phi_{\nu\mu_2\ldots\mu_s)}&=0,\\
\delta\phi_{\mu_1\ldots\mu_s}&=\partial_{(\mu_1}\xi_{\mu_2\ldots\mu_s)},\\
\Gamma^{\nu}\Gamma^{\rho}\Gamma^{\lambda}\phi_{\nu\rho\lambda\mu_4\ldots\mu_s}&=0,\qquad\qquad \Gamma^{\nu}\xi_{\nu\mu_2\ldots\mu_{s-1}}=0,
\end{aligned} \tag{4.21}$$

[6] Γ_μ are the Clifford algebra generators and satisfy $\Gamma_\mu\Gamma_\nu+\Gamma_\nu\Gamma_\mu=2\eta_{\mu\nu}$, $\not\partial\equiv\Gamma^\mu\partial_\mu$. Γ-trace is a contraction of a spinor index and one tensor index with Γ_μ, e.g., $\Gamma_\nu\phi^{\mu\nu}\equiv\Gamma^{\alpha}_{\nu\,\beta}\phi^{\beta;\mu\nu}$.

[7] Obtained from the Lagrangian, equations of [54] have the form $G_{\mu_1\ldots\mu_s}-\frac{s}{2}\Gamma_{(\mu_1}\Gamma^{\nu}G_{\nu\mu_2\ldots\mu_s)}-\frac{s(s-1)}{4}\eta_{(\mu_1\mu_2}G^{\nu}{}_{\nu\mu_3\ldots\mu_s)}=0$, $G_{\mu_1\ldots\mu_s}=\not\partial\phi_{\mu_1\ldots\mu_s}-s\partial_{(\mu_1}\Gamma^{\nu}\phi_{\nu\mu_2\ldots\mu_s)}=0$, which is equivalent to (4.22).

where in order to get an *off-shell* gauge invariance the irreducibility of $\phi_{\alpha;(\mu_1\ldots\mu_s)}$ has to be relaxed to the triple Γ-tracelessness.

A massless totally anti-symmetric spin-$\mathbf{Y}\{(1,p)\}_{\frac{1}{2}}$ field can be described *off-shell* by a totally anti-symmetric spin-tensor field $\omega_{\alpha;[\mu_1\ldots\mu_p]}$ subjected to

$$\begin{aligned} \not{\partial}\omega_{\mu_1\ldots\mu_p} - p\partial_{[\mu_1}\Gamma^\nu\omega_{\nu\mu_2\ldots\mu_p]} &= 0, \\ \delta\omega_{\mu_1\ldots\mu_p} &= \partial_{[\mu_1}\xi_{\mu_2\ldots\mu_p]}, \end{aligned} \tag{4.22}$$

Similar to the bosonic case, (4.22) possesses reducible gauge transformations. The difference is that one can impose the triple Γ-tracelessness on $\omega_{\alpha;[\mu_1\ldots\mu_p]}$ but this restricts the first level gauge parameter to be Γ-traceless, $\Gamma^\nu\xi_{\nu\mu_2\ldots\mu_{p-1}} = 0$, and, hence, the second order gauge parameter has to be *on-mass-shell*, i.e., $\not{\partial}\xi_{\mu_1\ldots\mu_{p-2}} = 0$. Therefore, for equations of motion to possess an *off-shell* gauge symmetry of all orders no Γ-trace conditions have to be imposed on field/gauge parameters.

In the general case of a massless spin-$\mathbf{Y} = \mathbf{Y}\{(s_1,p_1),\ldots,(s_N,p_N)\}_{\frac{1}{2}}$ *field*, the pattern of gauge symmetries is given by the spin-tensors with the tensor part described by (4.19), the definition of the Weyl tensor remains unchanged also.

The descriptions based on tensor field $\phi_{\mu_1(s_1),\mu_2(s_2),\ldots,\mu_n(s_n)}$, which is analogous to the metric $g_{\mu\nu}$, are referred to as *metric-like*. At least in writing $\Box \equiv \partial_\mu\partial_\nu\eta^{\mu\nu}$ the explicit use of metric $\eta^{\mu\nu}$ is made of, which complicates the issue of introducing interactions with gravitation.

Let us note that in principle one can verify the gauge invariance of the field equations for massless mixed-symmetry fields despite the fact that the very form of gauge transformation is cumbersome due to Young symmetrizers. The advantage of the unfolded approach is in that gauge invariance at all levels of reducibility is manifest.

5. Unfolding dynamics and σ_- cohomology

In this section we, first, recall the definition of unfolding and the relation of the simplest unfolded systems to Lie algebras/modules and Chevalley-Eilenberg cohomology with coefficients. Second, peculiar properties of unfolded systems that describe free *fields* and specifically the so-called σ_- cohomology concept are recalled. Third, the very procedure of constructing the unfolded form is illustrated on the examples of massless spin-zero, spin-one, arbitrary totally symmetric spin-s and spin-$(s+\frac{1}{2})$ fields, the relation with the general statement of section 3 is pointed out in each of the examples.

5.1 General features

Some dynamical system is said to be unfolded [31–33] if it has the form

$$dW^{\mathcal{A}} = F^{\mathcal{A}}(W), \tag{5.1}$$

where $W^{\mathcal{A}}$ is a set[8] of differential forms of degree-$q_{\mathcal{A}}$ on some d-dimensional manifold $\mathcal{M}_d$, d - exterior differential on $\mathcal{M}_d$ and $F^{\mathcal{A}}(W)$ is an arbitrary degree-$(q_{\mathcal{A}}+1)$ function of $W^{\mathcal{A}}$ assumed to be expandable in terms of exterior (wedge) products only[9]

$$F^{\mathcal{A}}(W) = \sum_{n=1}^{\infty} \sum_{q_{\mathcal{B}_1}+\dots+q_{\mathcal{B}_n}=q_{\mathcal{A}}+1} f^{\mathcal{A}}{}_{\mathcal{B}_1\dots\mathcal{B}_n} W^{\mathcal{B}_1}\wedge\dots\wedge W^{\mathcal{B}_n}, \tag{5.2}$$

where $f^{\mathcal{A}}{}_{\mathcal{B}_1\dots\mathcal{B}_n}$ are constant coefficients satisfying $f^{\mathcal{A}}{}_{\mathcal{B}_1\dots\mathcal{B}_i\mathcal{B}_j\dots\mathcal{B}_n} = (-)^{q_{\mathcal{B}_i}q_{\mathcal{B}_j}} f^{\mathcal{A}}{}_{\mathcal{B}_1\dots\mathcal{B}_j\mathcal{B}_i\dots\mathcal{B}_n}$. Moreover, $F^{\mathcal{A}}(W)$ must satisfy the integrability condition (called generalized Jacobi identity) obtained by applying d to (5.1)

$$F^{\mathcal{B}}\frac{\delta F^{\mathcal{A}}}{\delta W^{\mathcal{B}}} \equiv 0. \tag{5.3}$$

Any solution of (5.3) defines a free differential algebra (FDA) [55 − 58]. If Jacobi identities (5.3) are satisfied irrespective of $\mathcal{M}_d$ dimension,[10] the free differential algebra is referred to as universal [10, 59]. It is the universal algebras only that will be considered further.

Equations (5.1) are invariant under gauge transformations

$$\delta W^{\mathcal{A}}_{\mathbf{q}_{\mathcal{A}}} = d\epsilon^{\mathcal{A}}_{\mathbf{q}_{\mathcal{A}}-\mathbf{1}} + \epsilon^{\mathcal{B}}_{\mathbf{q}_{\mathcal{B}}-\mathbf{1}}\frac{\delta F^{\mathcal{A}}}{\delta W^{\mathcal{B}}}, \qquad\qquad \text{for} \qquad q_{\mathcal{A}}>0, \tag{5.4}$$

$$\delta W^{\mathcal{A}}_{\mathbf{0}} = \epsilon^{\mathcal{B}'}_{\mathbf{0}}\frac{\delta F^{\mathcal{A}}}{\delta W^{\mathcal{B}'}_{\mathbf{1}}}, \qquad \mathcal{B}' : q_{\mathcal{B}'}=1, \qquad \text{for} \qquad q_{\mathcal{A}}=0, \tag{5.5}$$

where $\epsilon^{\mathcal{A}}_{\mathbf{q}_{\mathcal{A}}-\mathbf{1}}$ is a degree-$(q_{\mathcal{A}}-1)$ form taking values in the same space as $W^{\mathcal{A}}_{\mathbf{q}_{\mathcal{A}}}$. In its turn, $\delta W^{\mathcal{A}}_{\mathbf{q}_{\mathcal{A}}} = 0$ can be treated as unfolded-like system for $\epsilon^{\mathcal{A}}_{\mathbf{q}_{\mathcal{A}}-\mathbf{1}}$, i.e., $d\epsilon^{\mathcal{A}}_{\mathbf{q}_{\mathcal{A}}-\mathbf{1}} = -\epsilon^{\mathcal{B}}_{\mathbf{q}_{\mathcal{B}}-\mathbf{1}}\frac{\delta F^{\mathcal{A}}}{\delta W^{\mathcal{B}}}$, there emerge second level gauge transformations

$$\delta\epsilon^{\mathcal{A}}_{\mathbf{q}_{\mathcal{A}}-\mathbf{1}} = d\xi^{\mathcal{A}}_{\mathbf{q}_{\mathcal{A}}-\mathbf{2}} - \xi^{\mathcal{B}}_{\mathbf{q}_{\mathcal{B}}-\mathbf{2}}\frac{\delta F^{\mathcal{A}}}{\delta W^{\mathcal{B}}}, \tag{5.6}$$

provided that $F^{\mathcal{A}}(W)$ is linear in matter fields and analogously for the gauge transformations at deeper levels. Therefore, the reducibility of gauge transformations is manifest in the unfolded approach. For a degree-$q_{\mathcal{A}}$ gauge field $W^{\mathcal{A}}_{\mathbf{q}_{\mathcal{A}}}$ there exist $q_{\mathcal{A}}$ levels of gauge transformations.

The use of the exterior algebra respects diffeomorphisms, which is very appropriate for introducing interactions with the gravitation. The whole information about the dynamics turns out to be contained in $F^{\mathcal{A}}(W)$ and one can even extend an unfolded system to other manifolds [59] simply by changing the exterior differential, the new unfolded system has literally the same form.

[8]In this section indices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are of arbitrary nature. In the cases of practical significance $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ vary over certain *irreps* of the Lorentz algebra.

[9]The wedge symbol $\wedge$ will be systematically omitted further.

[10]As the forms with the rank greater than d are identically zero, there exist certain identities, e.g., $W_{\mathbf{n}}\wedge W_{\mathbf{n}} \equiv 0$ for $n+m>d$, which make the operator $\frac{\delta}{\delta W^{\mathcal{A}}}$ to be ill-behaved.

Note that introducing enough auxiliary fields it is possible to reformulate any dynamical system in the unfolded form, although it may be difficult to unfold some particular system or to find all unfolded forms.

Collected below are some important cases of unfolded systems, which have a direct bearing on Lie algebras [55, 59].

Lie algebras/Flat connections. Let $\Omega^I \equiv \Omega^I_\mu dx^\mu$ be a subsector of degree-one forms. The only self-closed unfolded equations are of the form

$$d\Omega^I = -f^I_{\mathrm{JK}}\Omega^J\Omega^K. \tag{5.7}$$

Generalized Jacobi identity (5.3) implies the Jacobi identity $f^I_{\mathrm{JK}}f^J_{\mathrm{LM}}\Omega^K\Omega^L\Omega^M \equiv 0$ for some Lie algebra $\mathfrak{g}$ with structure constants f^I_{JK}. Therefore, the closed subsector of one-forms is in one-to-one correspondence with Lie algebras and (5.7) is the flatness condition for a connection Ω^I of $\mathfrak{g}$. This provides a coordinate-independent framework for describing background geometry. In the cases of interest, $\mathfrak{g}$ is $\mathfrak{iso}(d-1,1)$, $\mathfrak{so}(d-1,2)$, $\mathfrak{so}(d,1)$ and $\mathfrak{sp}(2n)$ [59]. Background geometry connection Ω^I is assumed to be of order zero, whereas all matter fields, including dynamical gravitation, are of the first order. All equations are assumed to be of the first order in matter fields and, hence, describe free fields only.

In this paper $\mathfrak{g} = \mathfrak{iso}(d-1,1)$ and $\Omega^I = \{\varpi^{a,b}, h^a\}$, where $\varpi^{a,b} \equiv \varpi^{a,b}_\mu dx^\mu$ is a Lorentz spin-connection and $h^a \equiv h^a_\mu dx^\mu$ is a background vielbein, which defines a non-holonomic basis of a tangent space at each point of the manifold. Flatness equation (5.7) for $\mathfrak{iso}(d-1,1)$-connection Ω^I reads

$$\begin{aligned} dh^a + \varpi^{a,}{}_b h^b &= 0, \\ d\varpi^{a,b} + \varpi^{a,}{}_c\varpi^{c,b} &= 0. \end{aligned} \tag{5.8}$$

The first is the zero torsion equation that expresses the Lorentz spin-connection via the first derivative of h^a_μ. The second can be recognized as the zero curvature equation. For example, in Cartesian coordinates the explicit solution is $\varpi^{a,b}_\mu = 0$ and $h^a_\mu = \delta^a_\mu$.

The advantage of description of background geometry as the flatness condition for a connection of the space-time symmetry algebra is in that this way is coordinate-independent. In what follows we assume $\varpi^{a,b}$, h^a to satisfy (5.8), which is enough if there is no need for the explicit form of the solution in some particular coordinate system. For instance, in (anti)-de Sitter space (5.8) is modified by the terms proportional to the cosmological constant λ^2

$$\begin{aligned} dh^a + \varpi^{a,}{}_b h^b &= 0, \\ d\varpi^{a,b} + \varpi^{a,}{}_c\varpi^{c,b} + \lambda^2 h^a h^b &= 0 \end{aligned} \tag{5.9}$$

and admits no simple solutions with $h^a_\mu = \delta^a_\mu$, $\varpi^{a,b} = 0$. Nevertheless, without giving the explicit solution, to imply that $\varpi^{a,b}$, h^a satisfy (5.9) is sufficient for general analysis, e.g., for constructing Lagrangians [60].

Contractible FDA. The simplest equations linear in matter fields of the form

$$dW^{\mathcal{A}}_{\mathbf{q}} = f^{\mathcal{A}}{}_{\mathcal{B}} W^{\mathcal{B}}_{\mathbf{q+1}} \tag{5.10}$$

can be reduced by linear transformations to either

$$dW^{\mathcal{A}}_{\mathbf{q}} = W^{\mathcal{A}}_{\mathbf{q+1}}, \qquad dW^{\mathcal{A}}_{\mathbf{q+1}} = 0, \tag{5.11}$$

or

$$dW^{\mathcal{A}}_{\mathbf{q}} = 0, \tag{5.12}$$

where the second equation of (5.11) is the consequence of Jacobi identities (5.3) for the first one. In the first case, by virtue of gauge transformations (5.4) $\delta W^{\mathcal{A}}_{\mathbf{q}} = d\xi^{\mathcal{A}}_{\mathbf{q-1}} + \chi^{\mathcal{A}}_{\mathbf{q}}$, $\delta W^{\mathcal{B}}_{\mathbf{q+1}} = d\chi^{\mathcal{A}}_{\mathbf{q}}$ the field $W^{\mathcal{A}}_{\mathbf{q}}$ can be gauged away. In both cases, by virtue of the Poincare's Lemma these equations are dynamically empty and correspond to the co-called contractible FDAs [55].

$\mathfrak{g}$-modules/Covariant constancy equations. Let $W^{\mathcal{A}}_{\mathbf{q}}$ be a closed subsector of q-forms of matter gauge fields. Linear in matter fields equations may involve the background $\mathfrak{g}$-connection Ω^I, which is of zeroth order. Such equations referred to as linearized over $\mathfrak{g}$ background (described by any solution Ω^I of (5.7)) have the form

$$dW^{\mathcal{A}}_{\mathbf{q}} = -\Omega^I f_I{}^{\mathcal{A}}{}_{\mathcal{B}} W^{\mathcal{B}}_{\mathbf{q}}. \tag{5.13}$$

Jacobi identity (5.3), where (5.7) is also taken into account,

$$\Omega^J \Omega^K \left(-f^I_{\mathrm{JK}} f_I{}^{\mathcal{A}}{}_{\mathcal{B}} + f_J{}^{\mathcal{A}}{}_{\mathcal{C}} f_K{}^{\mathcal{C}}{}_{\mathcal{B}}\right) W^{\mathcal{B}}_{\mathbf{q}} = 0 \tag{5.14}$$

implies $f_I{}^{\mathcal{A}}{}_{\mathcal{B}}$ to realize a representation of $\mathfrak{g}$. Therefore, the closed subsector of forms of definite degree is in one-to-one correspondence with $\mathfrak{g}$-modules, whereas

$$D_\Omega W^{\mathcal{A}}_{\mathbf{q}} \equiv dW^{\mathcal{A}}_{\mathbf{q}} + \Omega^I f_I{}^{\mathcal{A}}{}_{\mathcal{B}} W^{\mathcal{B}}_{\mathbf{q}} = 0 \tag{5.15}$$

is a covariant constancy equation and $(D_\Omega)^2 = 0$ since the connection is flat (5.7). In the cases of interest, $\mathcal{A}$ runs over certain finite-dimensional $\mathfrak{so}(d-1,1)$-*irreps*, i.e., $\mathfrak{g}$-modules decompose with respect to its subalgebra $\mathfrak{so}(d-1,1) \subset \mathfrak{g}$ into a direct sum of certain irreducible tensors. This is why we single out the Lorentz-covariant derivative D_L from the whole $\mathfrak{g}$-covariant derivative D_Ω, the remaining part acts vertically (algebraically). In the case of Ω^I being an $\mathfrak{iso}(d-1,1)$ flat connection, the Lorentz covariant derivative $D_L = d + \varpi$ satisfy ${D_L}^2 = 0$, as the exterior differential d does.

$$D_L T^{ab\ldots} = dT^{ab\ldots} + \varpi^{a,}{}_{c} T^{cb\ldots} + \varpi^{b,}{}_{c} T^{ac\ldots} + \cdots \tag{5.16}$$

In Cartesian coordinates $D_L \equiv d$ and we do not make any distinction between d and D_L, whereas in (anti)-de Sitter $(D_L)^2 \neq 0$ and the difference between d and D_L has to be taken into account. Also, zero torsion equation (5.8.1) can be rewritten as $D_L h^a = 0$. The remaining part of Ω^I, which acts algebraically and mixes different $\mathfrak{so}(d-1,1)$-modules into $\mathfrak{g}$-modules is associated with the generators of translations with gauge field h^a.

Gluing $\mathfrak{g}$-modules/Chevalley-Eilenberg cohomology. Let $W_{\mathbf{p}}$, $W_{\mathbf{q}}$ and $W_{\mathbf{r}}$ take values in $\mathfrak{g}$-modules $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$, the representations are realized by operators T_1, T_2 and T_3, respectively. Still linear in matter fields but nonlinear in background connection Ω^I equations are of the form

$$\begin{aligned} &D_\Omega W_{\mathbf{p}} \equiv dW_{\mathbf{p}} + T_1(\Omega)W_{\mathbf{p}} = f_{12}(\Omega,\ldots,\Omega)W_{\mathbf{q}},\\ &D_\Omega W_{\mathbf{q}} \equiv dW_{\mathbf{q}} + T_2(\Omega)W_{\mathbf{q}} = f_{23}(\Omega,\ldots,\Omega)W_{\mathbf{r}},\\ &D_\Omega W_{\mathbf{r}} \equiv \ldots, \end{aligned} \tag{5.17}$$

where the terms forming $\mathfrak{g}$-covariant derivative are isolated on the l.h.s. The two $\mathfrak{g}$-modules $\mathcal{R}_1$, $\mathcal{R}_2$ appear to be glued together by the term $f_{12}(\Omega,\ldots,\Omega) \in \mathrm{Hom}(\Lambda^{p-q+1}(\mathfrak{g})\otimes\mathcal{R}_2,\mathcal{R}_1)$. Jacobi identity (5.3) implies $f_{12}(\Omega,\ldots,\Omega)$ to be a Chevalley-Eilenberg cocycle with coefficients in $\mathcal{R}_2^*\otimes\mathcal{R}_1$, where $\mathcal{R}_2^*$ is a module contragradient to $\mathcal{R}_2$, and $f_{12}(\Omega,\ldots,\Omega)f_{23}(\Omega,\ldots,\Omega) = 0$. Coboundaries can be proved to be dynamically empty and can be removed by a field redefinition. Consequently, $f_{12}(\Omega,\ldots,\Omega)$ should be a nontrivial representative of the Chevalley-Eilenberg cohomology group with coefficients in $\mathcal{R}_2^*\otimes\mathcal{R}_1$. It follows also that nothing but zero forms can be joined to zero forms, i.e., the only possible linearized unfolded equations on forms of zero degree are (5.15) with $q=0$.

For any dynamical system, linearized over certain $\mathfrak{g}$-background (described by any solution Ω^I of (5.7)), equations of motion (5.15) along with gauge transformations of all orders of reducibility acquire a very simple form

$$\begin{aligned} \delta\xi_{\mathbf{1}} &= D_\Omega\xi_{\mathbf{0}},\\ &\ldots,\\ \delta\xi_{\mathbf{p-1}} &= D_\Omega\xi_{\mathbf{p-2}},\\ \delta W_{\mathbf{p}} &= D_\Omega\xi_{\mathbf{p-1}},\\ D_\Omega W_{\mathbf{p}} &= 0, \end{aligned} \tag{5.18}$$

where $W_{\mathbf{p}}$ takes values in certain $\mathfrak{g}$-module $\mathcal{R}$, D_Ω is the associated $\mathfrak{g}$-covariant derivative and $\xi_{\mathbf{p-k}}$, $k\in[1,p]$ are the gauge parameters of k-th order of reducibility taking values in the same $\mathfrak{g}$-module $\mathcal{R}$ and being forms of degree $(p-k)$. The gauge invariance at each order of reducibility and of equations of motion is due to $(D_\Omega)^2=0$. In some sense the gauge fields $W_{\mathbf{p}}$ and the gauge parameters $\xi_{\mathbf{p-k}}$ seem to play the same role at the linearized level. This uniformity is of essential importance when analyzing unfolded systems.

In the presence of gluing terms, e.g., when only two modules, say $\mathcal{R}_1$ and $\mathcal{R}_2$, are glued by nontrivial Chevalley-Eilenberg cocycle $f_{\mathrm{pr}}(\Omega,\ldots,\Omega)$ full chain of equations and gauge transformations (5.18) is modified to[11]

$$\begin{aligned}
\delta\xi^1_{\mathbf{1}} &= D_\Omega \xi^1_{\mathbf{0}}, \\
&\dots, \\
\delta\xi^1_{\mathbf{p-r}} &= D_\Omega \xi^1_{\mathbf{p-r-1}}, \\
\delta\xi^1_{\mathbf{p-r+1}} &= D_\Omega \xi^1_{\mathbf{p-r}} + f_{\mathrm{pr}}(\Omega,\dots,\Omega)\xi^2_{\mathbf{0}}, & \delta\xi^2_{\mathbf{1}} &= D_\Omega \xi^2_{\mathbf{0}}, \\
&\dots, & &\dots, \\
\delta\xi^1_{\mathbf{p-1}} &= D_\Omega \xi^1_{\mathbf{p-2}} + f_{\mathrm{pr}}(\Omega,\dots,\Omega)\xi^2_{\mathbf{r-2}}, & \delta\xi^2_{\mathbf{r-1}} &= D_\Omega \xi^2_{\mathbf{r-2}}, \\
\delta W^1_{\mathbf{p}} &= D_\Omega \xi_{\mathbf{p-1}} + f_{\mathrm{pr}}(\Omega,\dots,\Omega)\xi^2_{\mathbf{r-1}}, & \delta W^2_{\mathbf{r}} &= D_\Omega \xi^2_{\mathbf{r-1}}, \\
D_\Omega W^1_{\mathbf{p}} &= f_{\mathrm{pr}}(\Omega,\dots,\Omega) W^2_{\mathbf{r}} & D_\Omega W^2_{\mathbf{r}} &= 0,
\end{aligned} \tag{5.19}$$

where $W^1_{\mathbf{p}}$, $\xi^1_{\mathbf{p-k}}$ and $W^2_{\mathbf{r}}$, $\xi^2_{\mathbf{r-m}}$ take values in $\mathfrak{g}$-modules $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Important is that the gauge fields/parameters taking values in $\mathcal{R}_2$ contribute to the r.h.s. of the equations/gauge transformations for the gauge fields/parameters taking values in $\mathcal{R}_1$ but for forms of degree greater than $(p-r)$. The above two systems are nothing but the specializations of (5.1), (5.4), (5.6).

Also, let us note that there is no strong reason to make any distinction between the terms linear in the background $\mathfrak{g}$-connection Ω^I, which correspond to $\mathfrak{g}$-modules, and the terms of higher order in Ω^I, which correspond to Chevalley-Eilenberg cocycles. Both terms can be combined into a single object, the generalized covariant-derivative, $\mathcal{D} = d+T(\Omega)+f(\Omega,\dots,\Omega) \equiv D_\Omega + f(\Omega,\dots,\Omega)$ with the property $\mathcal{D}^2=0$. Moreover, by means of $\mathcal{D}$ (5.19) and (5.18) can be rewritten in a similar manner. Consequently, at the linearized level the most general unfolded equations and gauge transformations have the form

$$\begin{aligned}
\delta\xi_{\mathbf{1}} &= \mathcal{D}\xi_{\mathbf{0}}, \\
&\dots, \\
\delta\xi_{\mathbf{p-1}} &= \mathcal{D}\xi_{\mathbf{p-2}}, \\
\delta W_{\mathbf{p}} &= \mathcal{D}\xi_{\mathbf{p-1}}, \\
\mathcal{D}W_{\mathbf{p}} &= 0,
\end{aligned} \tag{5.20}$$

where $\mathcal{D}$ is built of d, background connection Ω^I and satisfies $\mathcal{D}^2 = 0$. The gauge fields/parameters take values in certain spaces $\mathcal{W}_{\mathbf{q}}$, viz., $W_{\mathbf{p}} \in \mathcal{W}_{\mathbf{p}}$, $\xi_{\mathbf{p-1}} \in \mathcal{W}_{\mathbf{p-1}}$,..., $\xi_{\mathbf{0}} \in \mathcal{W}_{\mathbf{0}}$. The elements of $\mathcal{W}_{\mathbf{q}}$ are differential forms with values in certain $\mathfrak{g}$-modules. It is also convenient to define higher degree spaces $\mathcal{W}_{\mathbf{q>p}}$: the equations $R_{\mathbf{p+1}} = \mathcal{D}W_{\mathbf{p}} = 0$ take values in $\mathcal{W}_{\mathbf{p+1}}$, $R_{\mathbf{p+2}} = \mathcal{D}R_{\mathbf{p+1}}$ belongs to $\mathcal{W}_{\mathbf{p+2}}$ and by virtue of $\mathcal{D}^2=0$ satisfies $R_{\mathbf{p+2}} \equiv 0$ and so on. Therefore, there exist certain identities, which belong to $\mathcal{W}_{\mathbf{p+2}}$, for the equations, which belong to $\mathcal{W}_{\mathbf{p+1}}$, and there exist certain identities, which belong to $\mathcal{W}_{\mathbf{p+3}}$ for the identities in $\mathcal{W}_{\mathbf{p+2}}$ and so on. At the field theoretical level these identities correspond to Bianchi identities for the first level gauge transformations, higher identities correspond to the Bianchi identities for the deeper levels of gauge transformations.

[11] The sign factor $(-)^{p-r+1}$ before $f_{\mathrm{pr}}(\Omega,\dots,\Omega)$ is ignored and is thought of as the part of the definition of $f_{\mathrm{pr}}(\Omega,\dots,\Omega)$.

A remark should be made that the forms belonging to $\mathcal{W}_{\mathbf{q}}$ may have different degrees if there are Chevalley-Eilenberg cocycles present. As a result, spaces $\mathcal{W}_{\mathbf{q}}$ may contain different number of elements. For instance, (5.19) can be reduced to (5.20) if one defines

$$\mathcal{W}_{\mathbf{q}} = \begin{cases} \{W^1_{\mathbf{q}}\}, & q < p - r, \\ \{W^1_{\mathbf{q}}, W^2_{\mathbf{q-p+r}}\}, & q \geq \cdot\ (p - r), \end{cases} \tag{5.21}$$

where $W^1_{\mathbf{q}}$ and $W^2_{\mathbf{q-p+r}}$ take values in $\mathfrak{g}$-modules $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. In terms of (5.19) the elements of $\mathcal{W}_{\mathbf{q}}$ for $q < (p-r)$ were referred to as $\xi^1_{\mathbf{q}}$, the elements of $\mathcal{W}_{\mathbf{q}}$ for $q = (p-r)\ldots(p-1)$ to as $\xi^1_{\mathbf{q}}$, $\xi^2_{\mathbf{q-p+r}}$, the elements of $\mathcal{W}_{\mathbf{q}}$ for $q = p$ to as $W^1_{\mathbf{p}}$, $W^2_{\mathbf{r}}$, the elements of $\mathcal{W}_{\mathbf{q}}$ for $q > p$ to as $R^1_{\mathbf{q}}$, $R^2_{\mathbf{q-p+r}}$ (the equations of motion and Bianchi identities).

5.2 Interpretation of unfolded systems describing free fields

When describing free fields, unfolded formulations will be referred to as *frame-like* [61, 60], though the very term *frame-like* has a more broad definition. These systems consist of equations (5.7), describing background geometry, of a finite chain of (5.17)-like equations, describing the gauge fields of the model and of (5.17)-like equations with $q = 0$ glued to gauge forms, i.e., it requires a Lie algebra $\mathfrak{g}$ (commonly $\mathfrak{iso}(d-1,1)$, $\mathfrak{so}(d-1,2)$ or $\mathfrak{so}(d,1)$), a set of $\mathfrak{g}$-modules $\mathcal{R}_0,\ldots,\mathcal{R}_N$ and an appropriate set of nontrivial Chevalley-Eilenberg cocycles $f_{0,1},\ldots,f_{N-1,N}$. The physical degrees of freedom are contained in the forms of zero degree and the module $\mathcal{R}_N$ in which zero degree forms take values is infinite-dimensional.

Since *frame-like* unfolded systems contain inevitably infinitely many fields, most of them being either auxiliary or *Stueckelberg*,[12] there arises a problem of reconstruction a *metric-like* formulation by a given unfolded formulation or of extracting the dynamical content for a given unfolded system.

The questions to be answered are: what are the dynamical fields, what are the differential gauge parameters and what are the gauge invariant equations of motion. These answers are not universal and depend on the chosen scheme of *interpretation*. Different *interpretations* correspond to *dual* descriptions of the same dynamical system.

Frame-like unfolded systems are endowed at the linearized level with additional structures, providing a natural way for *interpretation*. Indices $\mathcal{A}, \mathcal{B}, \ldots$ vary over certain finite dimensional Lorentz *irreps*, i.e., each of $\mathcal{R}_n$, $n = 0, \ldots, N$ decomposes as $\mathcal{R}_n = \sum_k \mathcal{P}_{n,k}$, where $\mathcal{P}_{n,k}$ are certain $\mathfrak{so}(d-1,1)$-modules characterized by $\mathbf{Y}_{\{n,k\}}$ Young diagrams of section 3.

We say that some *frame-like* unfolded system is given an *interpretation* if [62]

1. On the whole space $\mathcal{W} = \bigoplus_{q=0}^{\infty} \mathcal{W}_{\mathbf{q}}$, where the matter gauge fields, gauge parameters, equations of motion and Bianchi identities take values, there exists a bounded from below grading $g = 0, 1, \ldots$, i.e., $\mathcal{W}_{\mathbf{q}} = \bigoplus_{g=0}^{\infty} \mathcal{W}^g_{\mathbf{q}}$. The homogeneous element of $\mathcal{W}^g_{\mathbf{q}}$ is a certain differential form with values in certain $\mathfrak{so}(d-1,1)$-*irrep* and is denoted as $W^g_{\mathbf{q}}$. In simplest cases the grade is just the rank of the $\mathfrak{so}(d-1,1)$-*irrep* the element of $\mathcal{W}^g_{\mathbf{q}}$ takes values in.

[12] A field is called *Stueckelberg* if it can be gauged away by pure algebraic symmetry.

2. The closed subsector of one-forms Ω^I describes a background geometry, viz., Minkowski or (anti)-de Sitter. The generalized covariant derivative $\mathcal{D}$ is to be divided into three parts, the last one not being necessary nontrivial

$$\mathcal{D} = D_L - \sigma_- + \sigma_+, \tag{5.22}$$

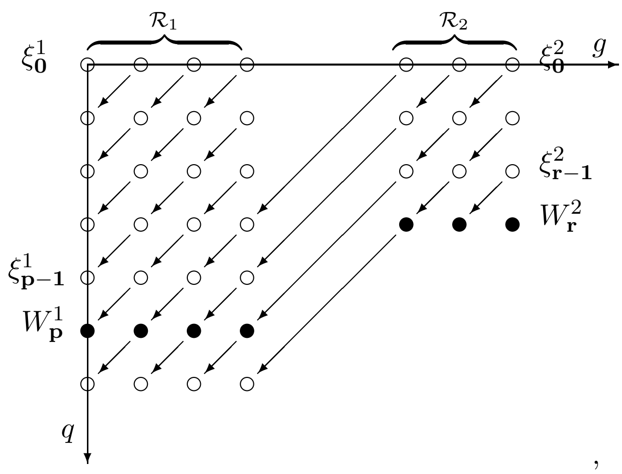
where D_L is a background Lorentz-covariant derivative and has a zero grade, σ_- is an operator of grade (-1) and σ_+ contains positive grade operators. The only differential part is in D_L, whereas $\sigma_\pm$ acts vertically and is built of the background vielbein h^a. If two modules are glued the gluing element is supposed to be of grade (-1) and also denoted as σ_-.

The equations then have the form

$$D_L W^n + \sum_{i=1}^{g-n} \sigma_+ \left(W^{n+i}\right) = \sigma_- \left(W^{n+1}\right), \qquad g = 0, 1, \dots \tag{5.23}$$

and analogously for the gauge transformations. In the flat space all operators of positive grade appear to be trivial for the massless case. As it does not affect the analysis let us consider a simplified version with $\sigma_+ = 0$. Then, $\mathcal{D}^2 = {D_L}^2 + \sigma_- D_L + D_L \sigma_- + (\sigma_-)^2 = 0$ is equivalent to ${D_L}^2 = 0$, $(\sigma_-)^2 = 0$ and $\sigma_- D_L + D_L \sigma_- = 0$. The first is a part of the flatness condition (5.8) for the $\mathfrak{iso}(d-1,1)$-connection. The second is the nilpotancy condition for σ_-. The third is satisfied provided that σ_- is twisted by the factor $(-)^{\Delta_g}$, where $(\Delta_g + 1)$ is a degree of σ_-, which is equal to the number of the vielbeins h^a that σ_- is built of. Indeed, the vielbeins h^a anticommute with D_L and, hence, the action of σ_- is to be twisted by the factor $(-)^{\Delta_g}$ in order $\sigma_- D_L + D_L \sigma_- = 0$ holds true. Without further mention, the pure sign factor $(-)^{\Delta_g}$ will be though of as a part of the definition of σ_-.

It is useful to illustrate the action of σ_- when, for example, unfolded equations on fields with values in two modules $\mathcal{R}_1$ and $\mathcal{R}_2$ are glued as in (5.19)



$$, \tag{5.24}$$

where short arrows stand for the action of σ_- within each module and long arrows for the action of σ_- between two modules (gluing terms) and dots stand for gauge fields/parameters at different grades, bold dots represent the elements of $\mathcal{W}_{\mathbf{p}} = \{W^1_{\mathbf{p}}, W^2_{\mathbf{r}}\}$.

All information about the dynamics turns out to be concealed in cohomology groups of σ_- [62], $\mathsf{H}(\sigma_-) = \frac{Ker(\sigma_-)}{Im(\sigma_-)}$. Let us analyze unfolded system (5.20), which at grade g has the form

$$\begin{aligned} \delta\xi^g_{\mathbf{1}} &= D_L\xi^g_{\mathbf{0}} + \sigma_-(\xi^{g+1}_{\mathbf{0}}), \\ &\dots, \\ \delta W^g_{\mathbf{p}} &= D_L\xi^g_{\mathbf{p-1}} + \sigma_-(\xi^{g+1}_{\mathbf{p-1}}), \\ 0 &= D_L W^g_{\mathbf{p}} + \sigma_-(W^{g+1}_{\mathbf{p}}), \end{aligned} \tag{5.25}$$

Beginning from the deepest level of gauge transformations and from the lowest grade, it is obvious that those $\xi^{g+1}_{\mathbf{0}}$ that are not σ_--closed can be treated as *Stueckelberg*(algebraic) gauge parameters for those $\xi^g_{\mathbf{1}}$ that belong to the image of σ_-. Therefore, those $\xi^g_{\mathbf{1}}$ that are σ_--exact can be gauged away. The leftover gauge symmetry satisfies $\delta\xi^g_{\mathbf{1}} = D_L\xi^g_{\mathbf{0}} + \sigma_-(\xi^{g+1}_{\mathbf{0}}) = 0$, so that those $\xi^{g+1}_{\mathbf{0}}$ that belong to the coimage of σ_- are expressed via derivative of $\xi^n_{\mathbf{0}}$. Having sieved $\mathcal{W}_{\mathbf{0}}$ and $\mathcal{W}_{\mathbf{1}}$ in this way, only those $\xi^g_{\mathbf{0}}$ are still independent that are σ_--closed and hence belong to $\mathsf{H}^0(\sigma_-)$, since only forms of zero-degree are elements of $\mathcal{W}_{\mathbf{0}}$. Then, those $\xi^{g+1}_{\mathbf{1}}$ that are not σ_--closed can be treated as *Stueckelberg* gauge parameters for those $\xi^g_{\mathbf{2}}$ that belong to the image of σ_-. Therefore, only those $\xi^g_{\mathbf{1}}$ are still independent that are σ_--closed but not σ_--exact and hence belong to $\mathsf{H}^1(\sigma_-)$. Having sieved $\mathcal{W}_{\mathbf{0}},\dots,\mathcal{W}_{\mathbf{p-1}}$ one after another, it turns out that independent differential gauge parameters at the k-th level are given by $\mathsf{H}^{p-k}(\sigma_-)$. Analogously, those fields $W^g_{\mathbf{p}}$ that are σ_--exact can be gauged away by virtue of *Stueckelberg* gauge parameters at $\mathcal{W}_{\mathbf{p-1}}$. Those fields $W^{g+1}_{\mathbf{p}}$ that are not σ_--closed can be expressed via derivatives of fields at lower grade by virtue of the equations $R^g_{\mathbf{p+1}} \equiv D_L W^g_{\mathbf{p}} + \sigma_-(W^{g+1}_{\mathbf{p}}) = 0$, these fields are called *auxiliary*. Therefore, the *dynamical* fields, i.e., those that are neither *auxiliary* nor *Stueckelberg*, are given by $\mathsf{H}^p(\sigma_-)$. The nilpotency of $\mathcal{D}^2 \equiv 0$ implies certain relations of the form $D_L R^g_{\mathbf{p+1}} + \sigma_-(R^{g+1}_{\mathbf{p+1}}) = 0$ between $R^g_{\mathbf{p+1}}$. Therefore, *auxiliary* fields are expressed by virtue of σ_--exact $R^g_{\mathbf{p+1}}$ and σ_--non-closed $R^{g+1}_{\mathbf{p+1}}$ are themselves expressed via derivatives of $R^g_{\mathbf{p+1}}$. Consequently, the independent equations on *dynamical* fields are given by $\mathsf{H}^{p+1}(\sigma_-)$. From the cohomological point of view higher degree forms correspond to certain nontrivial relations between equations called Bianchi identities and manifest gauge nature of equations. As there are generally more than one levels of gauge transformations, one can expect the higher cohomological group to be nontrivial.

To sum up, the field-theoretical meaning of the σ_- cohomology groups is collected in table 1.

To distinguish fields in cohomological sense the following convention is introduced [63]:

Stueckelberg fields(gauge parameters) are those that can be eliminated by pure algebraic symmetry, i.e., these fields are σ_--exact (are those that can be used to gauge away certain fields, i.e., these parameters are not σ_--closed).

Auxiliary fields are those that can be expressed via derivatives of some other fields, i.e., these fields are σ_--nonclosed. Let us note that this definition is generally ambiguous,

cohomology group	interpretation
$\mathsf{H}^{p-k},\ k=1\dots p$	differential gauge parameters at the k-th level of reducibility
H^{p}	*dynamical* fields
H^{p+1}	independent gauge invariant equations on *dynamical* fields
$\mathsf{H}^{p+k+1},\ k=1\dots p$	Bianchi identities for the k-th order gauge symmetry
$\mathsf{H}^{p+k},\ k>p$	supposed to be trivial in a regular case

Table 1: Field-theoretical meaning of the σ_- cohomology groups.

e.g., in system

$$\begin{aligned}\partial A &= B,\\ \partial B &= A,\end{aligned} \tag{5.26}$$

which field is *auxiliary* is a matter of choice. This ambiguity is removed by virtue of grade [62, 63].

Dynamical fields are those that are neither *auxiliary* nor *Stueckelberg*, i.e., these fields are representatives of σ_--cohomology classes.

Note that if certain *dynamical* field given by H^p belongs to the grade-g_f subspace $\mathcal{W}^{g_f}_{\mathbf{p}}$ and the corresponding equations given by H^{p+1} belong to the grade-g_e subspace $\mathcal{W}^{g_e}_{\mathbf{p+1}}$, the order of equations is equal to g_e-g_f+1.

If the unfolded system is supposed to admit a lagrangian formulation, there has to be a one-to-one correspondence between the number of *dynamical* fields and that of equations, k-th level gauge symmetries and k-th level Bianchi identities, i.e., in a certain sense there should be a duality $\mathsf{H}^{p-k}\sim\mathsf{H}^{p+1+k}$, $k\in[0,p]$. This takes place for all unfolded systems exemplified in this paper.

Since the operators involved, i.e., D_L and σ_-, are of grade 0 and -1 only, the fields with the grade greater than g_0, for any $g_0\geq 0$, form a quotient module, thus describing the same system on its own - *dual* formulations. This picture partly breaks in $(A)dS_d$ because of appearance of grade $+1$ nilpotent operator σ_+.

$$D=D_L+\sigma_-+\lambda^2\sigma_+ \tag{5.27}$$

The specific character of *frame-like* unfolded systems, which will be crucial for us, is that by virtue of the inverse background vielbein h^μ_a all form indices of any gauge field/parameter $W^{a_1(s_1),\dots,a_p(s_m)}_{\mathbf{q}}$ can be converted to the fiber ones

$$W^{a_1(s_1),\dots,a_m(s_m)|[d_1\dots d_q]}=W^{a_1(s_1),\dots,a_m(s_m)}_{\mu_1\dots\mu_q}h^{\mu_1 d_1}\dots h^{\mu_q d_q}. \tag{5.28}$$

The resulting tensor $W^{a_1(s_1),\dots,a_m(s_m)|[d_1\dots d_q]}$ is not irreducible and can be decomposed into $\mathfrak{so}(d-1,1)$-*irreps* according to the $\mathfrak{so}(d-1,1)$-tensor product rule

$$W^{a_1(s_1),\dots,a_m(s_m)|[d_1\dots d_q]}\Longrightarrow\mathbf{Y}\{s_1,\dots,s_m\}\bigotimes_{\mathfrak{so}(d-1,1)}\mathbf{Y}\{(1,q)\}=\bigoplus_\alpha\mathbf{Y}_\alpha, \tag{5.29}$$

where one-column Young diagram $\mathbf{Y}\{(1,q)\}$ of height-q represents the anti-symmetric indices $d_1,\dots,d_q$ and $\mathbf{Y}_\alpha$ are the $\mathfrak{so}(d-1,1)$-*irreps* the tensor product decomposes into. Having converted all fields (parameters, equations) of a given unfolded system to the fiber tensors one can make the identification of the fields in the unfolded approach and those of the *metric-like* approach, though the general covariance, gauge invariance and other advantages of the unfolded formalism will be not manifest. While constructing the unfolded formulations of a known *metric-like* systems or interpreting a given unfolded system in terms of *metric-like* fields, this technique will be widely used. Moreover, the calculation of σ_- cohomology groups is largely based on the explicit evaluation of (5.29)-like expressions. Once an unfolded form is known no use of this technique is needed either to generalize it to other backgrounds or to introduce interactions.

Field $W^{a_1(s_1),\dots,a_m(s_m)|[d_1\dots d_q]}$ is traceless in $a_1,\dots,a_m$ and $d_1,\dots,d_q$, separately. However, the cross traces need not vanish and, hence, some of $\mathbf{Y}_\alpha$ represent traces. To distinguish between the traces of different orders let us introduce the following: the $\mathbf{Y}$-valued degree-q form $W^{\mathbf{Y}}_{\mathbf{q}}$ is said to be a trace of the r-th order iff it has the form

$$W^{\mathbf{Y}}_{\mathbf{q}} = \underbrace{h\dots h}_{r}\underbrace{h\dots h}_{q-r}C^{\mathbf{Y}'}_{\mathbf{0}} \tag{5.30}$$

for some $\mathbf{Y}'$-valued degree-0 form $C^{\mathbf{Y}'}_{\mathbf{0}}$, where $(q-r)$ indices of the background vielbeins h^a are contracted with the tensor representing $\mathbf{Y}'$ and r indices are free for the whole expression to take values in $\mathbf{Y}$, the appropriate Young symmetrizer is implied. When the indices of $W^{\mathbf{Y}}_{\mathbf{q}}$ are converted to the fiber ones, the expression takes the form $\underbrace{\eta^{\cdots}\dots\eta^{\cdots}}_{r}C^{\mathbf{Y}'}$, i.e., $C^{\mathbf{Y}'}$ represents a trace of the r-th order.

Different aspects of unfolding are illustrated on the examples of a massless scalar field, a massless spin-one field and a massless spin-s and spin-$(s+\frac{1}{2})$ fields.

5.3 Examples of unfolding

Example 1. ***Unfolding a scalar field [31, 62].*** *First, it should be noted that it is easy, of course, to convert the Klein-Gordon equation $\Box C = 0$ to a system of first order equations*

$$\begin{aligned} \partial_\mu C &= C_\mu, \\ \partial^\mu C_\mu &= 0, \end{aligned} \tag{5.31}$$

but when writing the second equation the explicit form of the metric is to be used, hence, these equations are not of unfolded form (5.1).

Described in terms of a scalar field $C(x)$, the theory is brought into a non gauge form, therefore, zero-forms only are allowed, or else there would be some gauge symmetry according to (5.4). The most general r.h.s. of $dC = \cdots$ has the form

$$dC = h_a C^a \tag{5.32}$$

for some vector-valued zero-form C^a. This equation just parameterizes the first derivative of $C(x)$ and is similar to the first of (5.31). There are three terms allowed on the r.h.s.

of $dC^a = \cdots$

$$dC^a = h_b C^{a,b} + h_b C^{ab} + h^a C' \tag{5.33}$$

with $C^{a,b}$, C^{ab} *and* C' *taking values in antisymmetric, symmetric and scalar* $\mathfrak{so}(d-1,1)$-irreps, *i.e.,* $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, $\square\!\square$, $\bullet$. *The first term is forbidden by Bianchi identity* $h_a dC^a \equiv 0$. *To analyze the remaining terms let us decompose* dC^a *into* $\mathfrak{so}(d-1,1)$-irreps

$$\partial_\mu C^a h^{\mu b} = \square \otimes \square = \square\!\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \bullet, \tag{5.34}$$

where $\begin{smallmatrix}\square\\\square\end{smallmatrix} = \partial^{[a}C^{b]}$, $\square\!\square = \partial^{(a}C^{a)} - \frac{1}{d}\eta^{aa}\partial_b C^b$, $\bullet = \partial_a C^a = \Box C$.

The $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ *component represents Bianchi identity in the first order reformulation of the Klein-Gordon equation. Indeed, expressing* C^a *as* $\partial^a C$ *implies* $\partial^{[a}C^{b]} \equiv 0$. $\bullet$ *represents the desired Klein-Gordon equation, whereas* $\square\!\square$ *is the only component allowed to be nonzero* on-mass-shell. *Therefore, to impose the Klein-Gordon equation the term* $h^a C'$ *has to be omitted,*

$$dC^a = h_b C^{ab}. \tag{5.35}$$

The only possible r.h.s. for $dC^{aa} = \cdots$ *compatible with Jacobi identity* $h_b dC^{ab} \equiv 0$ *is of the form* $dC^{aa} = h_b C^{aab}$ *for* C^{aaa} *taking values in a rank-three symmetric* $\mathfrak{so}(d-1,1)$-irrep, *i.e.,* $\square\!\square\!\square$. *The process of unfolding continues unambiguously in this way and results in the full system*

$$dC^{a(k)} = h_b C^{a(k)b} \equiv F^{a(k)}(C), \qquad k = 0, 1, \ldots, \tag{5.36}$$

where $C^{a(k)}$ *is rank-k totally symmetric traceless field, i.e., taking values in* $\boxed{\quad k \quad}$ $\mathfrak{so}(d-1,1)$-irrep.

To make a connection with the general statement of section 3, for $\mathbf{Y} = \mathbf{Y}\{(0,0)\}$ *the general scheme results in*

$$\begin{array}{rccccc}
\mathbf{Y}_g: & \bullet & \square & \square\!\square & \square\!\square\!\square & \ldots \\
\{n,k\}: & \{0,0\} & \{0,1\} & \{0,2\} & \{0,3\} & \ldots \\
g: & g=0 & g=1 & g=2 & g=3 & \ldots \\
q_g: & q_0=0 & q_1=0 & q_2=0 & q_3=0 & \ldots
\end{array}$$

therefore, spaces $\mathcal{W}_\mathbf{q}$ *and operator* σ_- *which enters* $\mathcal{D} = D_L - \sigma_-$ *should be defined as*

$$\begin{aligned}
\mathcal{W}_\mathbf{q} &= \{W_\mathbf{q}, W^a_\mathbf{q}, W^{aa}_\mathbf{q}, \ldots\}, \\
\sigma_-(W^g_\mathbf{q}) &= \begin{cases} 0, & g = 0, \\ h_b W^{a(g-1)b}_\mathbf{q}, & g > 0. \end{cases}
\end{aligned} \tag{5.37}$$

Consequently, $\mathcal{W}$ *is a space of various differential forms with values in totally symmetric traceless* $\mathfrak{so}(d-1,1)$*-tensors, graded by the form degree and by the rank of* $\mathfrak{so}(d-1,1)$-irreps. *Space* $\mathcal{W}_\mathbf{q}$ *forms an irreducible* $\mathfrak{iso}(d-1,1)$*-module. System (5.36) reads*

$$\mathcal{D}\omega_\mathbf{0} = 0, \qquad \omega_\mathbf{0} \in \mathcal{W}_\mathbf{0}. \tag{5.38}$$

The cohomology groups of σ_- *are easy to find:*

$\mathsf{H}^0(\sigma_-) = C(x)$**,** *as the lowest grade field is automatically closed, $\sigma_-(C) = 0$, and it cannot be exact as a form of zero degree. Contrariwise, zero-forms with grade greater than zero are not closed, $\sigma_-(C^{k+1}) = h_b C^{a(k)b} = 0 \Longleftrightarrow C^{k+1} = 0$;*

$\mathsf{H}^1(\sigma_-) = h^a B(x)_{\mathbf{0}}$**,** *for some $B(x)_{\mathbf{0}} \in \mathcal{W}^0_{\mathbf{0}}$. One-form $B^a_{\mathbf{1}} = h^a B(x)_{\mathbf{0}}$ is closed, $\sigma_-(B^a_{\mathbf{1}}) = h_a B^a_{\mathbf{1}} = h_a h^a B(x)_{\mathbf{0}} \equiv 0$, and it cannot be represented as $\sigma_-(C^2) = h_b C^{ab}$ as far as C^{ab} is traceless. The cohomology groups at higher grade are trivial since $\sigma_-(h^a B^{a(k)}(x)_{\mathbf{0}}) \neq 0$ for $k > 0$;*

$\mathsf{H}^{q>1}(\sigma_-) = \emptyset$**,** *as one can make sure.*

The interpretation *in terms of section 5.2 is as follows: as is expected, the* dynamical *field given by H^0 is $C(x)$. The equations are given by the projection of $dC^a = h_b C^{ab}$ to H^1, i.e., $\Box C = 0$. The fields with $k > 0$ are* auxiliary*, being expressed via derivatives of $C(x)$, $C^{a(k)} = \partial^a \ldots \partial^a C(x)$. Inasmuch as there is no gauge symmetry in the system, the higher cohomology groups are trivial.*

An infinitely many dual *formulations of a massless scalar* field *are also included. Indeed, the fields with $k \geq k_0 > 0$ form a quotient module $\mathcal{R}_{k_0}$, i.e., one can consistently write*

$$dC^k = \sigma_-(C^{k+1}), \quad k \geq k_0. \tag{5.39}$$

The first equation

$$dC^{a(k_0)} = h_b C^{a(k_0)b} \tag{5.40}$$

imposes on the dynamical *field $C^{a(k_0)}$*

$$\begin{aligned} &\partial_b C^{ba(k_0-1)} = 0, \\ &\partial^{[b} C^{c]a(k_0-1)} = 0, \end{aligned} \tag{5.41}$$

which imply, at least locally, $C^{a(k_0)} = \overbrace{\partial^a \ldots \partial^a}^{k_0} C$ and $\Box C = 0$, for some field $C(x)$. All fields at higher grade are auxiliary. *This statement is confirmed by the cohomology analysis: $\mathsf{H}^0(\mathcal{R}_{k_0}, \sigma_-) = C^{k_0}$, i.e., scalar is described in terms of a traceless rank-k_0 symmetric tensor field subjected to the equations given by $\mathsf{H}^1(\mathcal{R}_{k_0}, \sigma_-) = h^a B^{a(k_0-1)}_{\mathbf{0}} - \frac{k_0(k_0-1)}{2(d+2k_0-4)} \eta^{aa} h_c B^{a(k_0-2)c}_{\mathbf{0}} + h_c B^{a(k_0),c}_{\mathbf{0}}$, for $B^{a(k_0-1)}_{\mathbf{0}}$, $B^{a(k_0),c}_{\mathbf{0}}$ taking values in $\mathfrak{so}(d-1,1)$*-irreps *characterized by diagrams $\boxed{\;k_0-1\;}$, $\begin{array}{l}\boxed{\quad k_0 \quad}\\ \boxed{\;}\end{array}$. These two elements represents just the components of $\mathcal{W}^{k_0}_{\mathbf{1}} \sim \boxed{\quad k_0 \quad} \otimes \boxed{\;}$ with the symmetry of $\boxed{\;k_0-1\;}$ (trace) and $\begin{array}{l}\boxed{\quad k_0 \quad}\\ \boxed{\;}\end{array}$, the third component with the symmetry of $\boxed{\;k_0+1\;}$ is exact.*

The above results can be easily extended to $(A)dS_d$ background and to a massive scalar field [62],

$$D_L C^k = \sigma_-(C^{k+1}) + \sigma_+(C^{k-1}) = 0, \tag{5.42}$$

where there appears a nontrivial positive grade operator

$$\sigma_+(C^{k-1}) = (m^2 - \lambda^2 k(k+d-1)) \left(h^a C^{a(k-1)} - \frac{(s-1)}{d+2k-4} \eta^{aa} h_b C^{a(k-1)b} \right) \tag{5.43}$$

JHEP07(2008)004

and D_L is a Lorentz-covariant derivative in $(A)dS_d$. In the $(A)dS_d$ case the situation is more interesting due to the appearance of σ_+ operator and the possibility of the accidental degeneracy of σ_+ when $m^2 = \lambda^2 k'(k'+d-1)$ for some k' [62]. ■

Example 2. ***Unfolding Maxwell equations [64, 65].*** *An example of massless spin-one* field *is more interesting as a gauge system. The main object is a one-form A_μ. From gauge transformation law*

$$\delta A_{\mathbf{1}} = d\xi_{\mathbf{0}} \tag{5.44}$$

it follows that there should not be other forms of rank greater than zero. If this were the case the gauge parameters of these new forms would be involved by virtue of (5.4), making the gauge law for $A_{\mathbf{1}}$ inappropriate. Indeed, there are two possibilities to introduce higher degree forms on the r.h.s. of $dA = \cdots$:

(i) $dA_{\mathbf{1}} = R_{\mathbf{2}}$ for a scalar valued degree-two form $R_{\mathbf{2}}$, which possesses its own gauge parameter $\chi_{\mathbf{1}}$, $\delta R_{\mathbf{2}} = d\chi_{\mathbf{1}}$. In accordance with (5.4) the gauge transformation law for $A_{\mathbf{1}}$ is modified to $\delta A_{\mathbf{1}} = d\xi_{\mathbf{0}} + \chi_{\mathbf{1}}$ so that $A_{\mathbf{1}}$ can be gauged away by virtue of Stueckelberg *gauge parameter $\chi_{\mathbf{1}}$. This case corresponds to contractible free differential algebras (5.10).*

(ii) $dA_{\mathbf{1}} = h_b\omega^b_{\mathbf{1}}$ for a vector valued degree-one form $\omega^a_{\mathbf{1}}$, which possesses its own gauge parameter $\epsilon^a_{\mathbf{0}}$.

In accordance with (5.4) the gauge transformation law for $A_{\mathbf{1}}$ is modified to $\delta A_{\mathbf{1}} = d\xi_{\mathbf{0}} + h_b\epsilon^b_{\mathbf{0}}$ so that $A_{\mathbf{1}}$ can be gauged away by virtue of Stueckelberg *gauge parameter $\epsilon^a_{\mathbf{0}}$. Consequently, in both cases $A_{\mathbf{1}}$ can be made non-dynamical, which is not what was expected. Therefore, the only possibility is to introduce a zero-form $C^{[ab]}_{\mathbf{0}}$, anti-symmetric in a, b, parameterizing by virtue of*

$$dA_{\mathbf{1}} = h_a h_b C^{[ab]}_{\mathbf{0}} \tag{5.45}$$

the Maxwell field strength. Bianchi identity $h_a h_b dC^{[ab]}_{\mathbf{0}} \equiv 0$ tolerates two terms on the r.h.s of $dC^{[ab]}_{\mathbf{0}} = \cdots$

$$dC^{[ab]}_{\mathbf{0}} = h_c C^{[ab],c}_{\mathbf{0}} + h^{[a}C^{b]}_{\mathbf{0}}, \tag{5.46}$$

with $C^{[ab],c}_{\mathbf{0}}$ and $C^a_{\mathbf{0}}$ taking values in $\ydiagram{2,1}$ and $\square$ $\mathfrak{so}(d-1,1)$-irreps, *the former taken in antisymmetric basis. In order to determine which of the terms should be omitted, if any, the decomposition of the first derivative of the Maxwell field strength into Lorentz* irreps *has to be analyzed*

$$C^{[ab]|c} \equiv \partial_\mu C^{[ab]} h^{\mu c} = \square \otimes \ydiagram{1,1} = \ydiagram{1,1,1} \oplus \square \oplus \ydiagram{2,1}, \tag{5.47}$$

where $\ydiagram{1,1,1} = \partial^{[a}C^{bc]}$, $\square = \partial_b C^{ab}$, and $\ydiagram{2,1} = \partial^c C^{[ab]} - \partial^{[c}C^{ab]} + \frac{2}{d-1}\eta^{c[a}\partial_d C^{b]d}$. $\partial^{[a}C^{bc]}$ is identically zero provided that $C^{[ab]}$ is expressed via the first derivative of A_μ, or represents the second pair of Maxwell equations in terms of field strength $C^{[ab]}$. In both cases, this component should be zero and, moreover, this can not be kept nonzero as there is *no $h_c C^{[abc]}$-like term allowed on the r.h.s. of (5.46). The second, $\partial_b C^{ab}$, is the desired Maxwell equations in terms of A_μ, or the first pair of Maxwell equations in terms of the field strength. Consequently, to impose Maxwell equations term $h^{[a}C^{b]}$ has to be omitted, whereas the third is the only component allowed to be nonzero* on-mass-shell.

Again, unfolding continues unambiguously and requires an infinite set of fields $C_{\mathbf{0}}^{[ab],c(k)}$ taking values[13] *in* k *to be introduced*

$$dC^{[ab],c(k)} = h_d C^{[ab],c(k)d}, \qquad k = 0, 1, \dots \tag{5.48}$$

The full system has the form

$$\begin{aligned} dA_{\mathbf{1}} &= h_a h_b C_{\mathbf{0}}^{a,b} \equiv F(A, C), & \delta A_{\mathbf{1}} &= d\xi_{\mathbf{0}}, \\ dC_{\mathbf{0}}^{[ab],c(k)} &= h_d C_{\mathbf{0}}^{[ab],c(k)d} \equiv F^{[ab],c(k)}(C), & k &= 0, 1, \dots \end{aligned} \tag{5.49}$$

and represents two modules being glued together by the term on the r.h.s of the first equation.

For $\mathbf{Y} = \mathbf{Y}\{(1,1)\}$, i.e., $N = 1$ and $p = 1$, the general scheme of section 3 results in

$$\begin{array}{rcccccc} \mathbf{Y}_g: & \bullet & & & & \dots \\ \{n,k\}: & \{1,0\} & \{0,0\} & \{0,1\} & \{0,2\} & \dots \\ g: & g=0 & g=1 & g=2 & g=3 & \dots \\ q_g: & q_0 = 1 & q_1 = 0 & q_2 = 0 & q_3 = 0 & \dots \end{array}$$

therefore,

$$\mathcal{W}_{\mathbf{q}} = \begin{cases} \{W_{\mathbf{0}}\}, & q = 0, \\ \{W_{\mathbf{q}}, W_{\mathbf{q-1}}^{[ab]}, W_{\mathbf{q-1}}^{[ab],c}, \dots\}, & q > 0, \end{cases} \tag{5.50}$$

let us point out the shortening of $\mathcal{W}_{\mathbf{0}}$. The action of σ_- on $W_{\mathbf{q}} \in \mathcal{W}_{\mathbf{q}}$ is defined as

$$\sigma_-(W_{\mathbf{q}}^g) = \begin{cases} 0, & g = 0, \\ h_a h_b W_{\mathbf{q-1}}^{[ab]}, & g = 1, \\ h_d W_{\mathbf{q-1}}^{[ab],c(g-2)d}, & g > 1. \end{cases} \tag{5.51}$$

With $\mathcal{D} = D_L - \sigma_-$, unfolded system (5.49) can be rewritten as

$$\mathcal{D}\omega_{\mathbf{1}} = 0, \qquad \delta\omega_{\mathbf{1}} = \mathcal{D}\xi_{\mathbf{0}}, \tag{5.52}$$

where $\omega_{\mathbf{1}} \in \mathcal{W}_{\mathbf{1}}$ and $\xi_{\mathbf{0}} \in \mathcal{W}_{\mathbf{0}}$. The fields at $\{n = 1, k = 0\} \sim g = 0$ form a finite-dimensional $\mathfrak{iso}(d-1,1)$-module, whereas the fields with $\{n = 0, k = 0, 1, \dots\} \sim g > 0$ form an infinite-dimensional $\mathfrak{iso}(d-1,1)$-module.

Again, the cohomology groups are easy to find: gauge parameters are given by $\mathsf{H}^0(\sigma_-) = \xi_{\mathbf{0}}$, dynamical *fields by* $\mathsf{H}^1(\sigma_-) = A_{\mathbf{1}}$, *Maxwell equations by* $\mathsf{H}^2(\sigma_-) = h^{[a}B_{\mathbf{0}}^{b]}$, *and the Bianchi identities for the first order gauge transformations by* $\mathsf{H}^3(\sigma_-) = h^a h^b B_{\mathbf{0}}$,

[13]The fields $C^{[ab],c(k)}$ are taken in antisymmetric basis, i.e., they are antisymmetric in a, b, symmetric in $c(k)$ and $C^{[ab,d]c(k-1)} \equiv 0$.

the rest of groups are empty $\mathsf{H}^{k>3}(\sigma_-) = \emptyset$. *Important is the direct correspondence between the number of fields in* H^1 *and the equations in* H^2, *the gauge parameters in* H^0 *and the Bianchi identities in* H^3.

There are infinitely many of non-gauge dual *formulations based on* $C^{[ab],c(k_0)}$, *e.g., for* $k_0 = 0$ *one recovers the Maxwell equations in terms of the field strength*

$$\begin{aligned} \partial^{[a}C^{b,c]} &= 0, \\ \partial_b C^{a,b} &= 0. \end{aligned} \tag{5.53}$$

$\blacksquare$

Let us note that the ambiguity at the first step of unfolding (the terms $h^a C'$ and $h^{[a}C^{b]}$ in the above two examples) expresses the possibility of introducing a mass term.

Example 3. ***Unfolding a totally symmetric massless spin-s field [31, 64, 66].*** *We start from the* metric-like *description of a massless spin-s* field *in terms of a double-traceless symmetric rank-s field* $\phi_{\mu_1\ldots\mu_s}$ *satisfying (4.13), the second order equations are invariant with respect to the first order gauge transformation with traceless rank-$(s-1)$ gauge parameter* $\xi_{\mu_1\ldots\mu_{s-1}}$. *Inasmuch as the Young diagram* $\boxed{\quad s \quad}$ *of* $\mathfrak{so}(d-2)$ *is of the height one, there are gauge transformations of the first level only. Therefore, the dynamical field in the unfolded approach has to be a one-form. There is only one way to identify* $\phi_{\mu_1\ldots\mu_s}$ *and* $\xi_{\mu_1\ldots\mu_{s-1}}$ *with certain* $\omega^{\mathbf{Y}_0}_{\mathbf{1}}$ *and* $\xi^{\mathbf{Y}_0}_{\mathbf{0}}$ *taking values in the same* $\mathfrak{so}(d-1,1)$-irrep $\mathbf{Y}_0$, *namely,* $\mathbf{Y}_0 = \boxed{\quad s-1 \quad}$, *i.e.,* $\omega^{a(s-1)}_{\mathbf{1}}$ *and* $\xi^{a(s-1)}_{\mathbf{0}}$ *takes values in a rank-$(s-1)$ symmetric traceless tensors. Field* $\phi_{\mu_1\ldots\mu_s}$ *is identified with a totally symmetric part of* $\omega^{a(s-1)}_{\mathbf{1}}$, *i.e.,* $\phi_{\mu_1\ldots\mu_s} = h^{b_1}_{(\mu_1}\ldots h^{b_{s-1}}_{\mu_{s-1}}\omega^{a_1\ldots a_{s-1}}_{\mu_s)}\eta_{a_1b_1}\ldots\eta_{a_{s-1}b_{s-1}}$, *the double-tracelessness condition is the consequence of the tracelessness of* $\omega^{a(s-1)}_{\mathbf{1}}$ *in* $a_1\ldots a_{s-1}$. $\xi_{\mu_1\ldots\mu_{s-1}}$ *is identified with* $\xi^{a(s-1)}_{\mathbf{0}}$ *directly* $\xi_{\mu_1\ldots\mu_{s-1}} = h^{b_1}_{\mu_1}\ldots h^{b_{s-1}}_{\mu_{s-1}}\xi^{a_1\ldots a_{s-1}}\eta_{a_1b_1}\ldots\eta_{a_{s-1}b_{s-1}}$ *and one can make sure that the Fronsdal's gauge transformation law is recovered from* $\delta\omega^{a(s-1)}_{\mathbf{1}} = d\xi^{a(s-1)}_{\mathbf{0}}$. *Field* $\omega^{a(s-1)}_{\mathbf{1}}$ *has a* redundant *component with the symmetry of* $\begin{array}{l}\boxed{\quad s-1 \quad}\\ \boxed{}\end{array}$, *which can be made* Stueckelberg *by virtue of gauge parameter* $\xi^{a(s-1),b}$ *with the same symmetry type, i.e.,*

$$\delta\omega^{a(s-1)}_{\mathbf{1}} = d\xi^{a(s-1)}_{\mathbf{0}} + h_c\xi^{a(s-1),c}_{\mathbf{0}}. \tag{5.54}$$

It automatically follows that there is a degree-one gauge field $\omega^{a(s-1),b}_{\mathbf{1}}$ *coupled with* $\omega^{a(s-1)}_{\mathbf{1}}$ *as*

$$d\omega^{a(s-1)}_{\mathbf{1}} = h_c\omega^{a(s-1),c}_{\mathbf{1}}. \tag{5.55}$$

The only solution to Bianchi identity $h_c d\omega^{a(s-1),c}_{\mathbf{1}} \equiv 0$ *is*[14]

$$d\omega^{a(s-1),b}_{\mathbf{1}} = h_c\omega^{a(s-1),bc}_{\mathbf{1}}, \tag{5.56}$$

where a new field with the symmetry of $\begin{array}{l}\boxed{\quad s-1 \quad}\\ \boxed{}\boxed{}\end{array}$ *is introduced, and so on*

$$d\omega^{a(s-1),b(k)}_{\mathbf{1}} = h_c\omega^{a(s-1),b(k)c}_{\mathbf{1}} \tag{5.57}$$

[14]The proof is not given as more general statement about the solutions of such equations is proved in the next section.

until the field with the symmetry of $\begin{array}{|c|}\hline \quad s-1 \quad \\\hline \\\hline\end{array}$*, for which the Bianchi identity solves as*

$$d\omega_{\mathbf{1}}^{a(s-1),b(s-1)} = h_c h_d C_{\mathbf{0}}^{a(s-1)c,b(s-1)d}, \tag{5.58}$$

where field $C_{\mathbf{0}}^{a(s),b(s)}$ *represents a generalized Weyl tensor and has the symmetry of* $\begin{array}{|c|}\hline \quad s \quad \\\hline \\\hline\end{array}$*. The solution of Bianchi identity* $h_c h_d dC_{\mathbf{0}}^{a(s-1)c,b(s-1)d} \equiv 0$ *has the form*

$$dC_{\mathbf{0}}^{a(s),b(s)} = h_c \left(C_{\mathbf{0}}^{a(s)c,b(s)} + \frac{s}{2} C_{\mathbf{0}}^{a(s)b,b(s-1)c} \right), \tag{5.59}$$

where $C_{\mathbf{0}}^{a(s+1),b(s)}$ *has the symmetry of* $\begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{s+1} \\\hline \quad s \quad & \multicolumn{1}{|c}{} \\\cline{1-1}\end{array}$*. The second term on the r.h.s. of (5.59) supplements the first one to have a proper Young symmetry. Further unfolding requires a set of degree-zero forms* $C_{\mathbf{0}}^{a(s+i),b(s)}$*, taking values in* $\mathfrak{so}(d-1,1)$-irreps *characterized by Young diagrams* $\begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{s+i} \\\hline \quad s \quad & \multicolumn{1}{|c}{} \\\cline{1-1}\end{array}$*. The full system has the form*

$$\begin{aligned}
d\omega_{\mathbf{1}}^{a(s-1),b(k)} &= h_c \omega_{\mathbf{1}}^{a(s-1),b(k)c}, \\
\delta\omega_{\mathbf{1}}^{a(s-1),b(k)} &= d\xi_{\mathbf{0}}^{a(s-1),b(k)} + h_c \xi_{\mathbf{0}}^{a(s-1),b(k)c}, & k \in [0, s-2], \\
d\omega_{\mathbf{1}}^{a(s-1),b(s-1)} &= h_c h_d C_{\mathbf{0}}^{a(s-1)c,b(s-1)d}, & \delta\omega_{\mathbf{1}}^{a(s-1),b(s-1)} = d\xi_{\mathbf{0}}^{a(s-1),b(s-1)} \\
dC^{a(s+i),b(s)} &= h_c \left(C^{a(s+i)c,b(s)} + \frac{s}{i+2} C^{a(s+i)b,b(s-1)c} \right), & i \in [0, \infty).
\end{aligned} \tag{5.60}$$

By the construction the system incorporates Fronsdal's field $\phi_{\mu_1 \ldots \mu_s}$ *with the correct gauge law, but there is still to be proved that the Fronsdal's equations are really imposed and that there are no other* dynamical *fields in the system. Let us reconstruct Fronsdal's equations (4.13), whereas the second statement will be proved as a part of a more general theorem. The first two equations of (5.60) read*

$$\partial_\mu \omega_\nu^{a(s-1)} - \partial_\nu \omega_\mu^{a(s-1)} = h_{c\mu} \omega_\nu^{a(s-1),c} - h_{c\nu} \omega_\mu^{a(s-1),c}, \tag{5.61}$$

$$\partial_\mu \omega_\nu^{a(s-1),b} - \partial_\nu \omega_\mu^{a(s-1),b} = h_{c\mu} \omega_\nu^{a(s-1),bc} - h_{c\nu} \omega_\mu^{a(s-1),bc} \tag{5.62}$$

It is convenient to convert all world indices to the fiber ones

$$\partial^c \omega^{a(s-1)|d} - \partial^d \omega^{a(s-1)|c} = \omega^{a(s-1),c|d} - \omega^{a(s-1),d|c}, \tag{5.63}$$

$$\partial^c \omega^{a(s-1),b|d} - \partial^d \omega^{a(s-1),b|c} = \omega^{a(s-1),bc|d} - \omega^{a(s-1),bd|c}, \tag{5.64}$$

where $\omega^{a(s-1)|b} \equiv \omega_\mu^{a(s-1)} h^{b\mu}$*,* $\omega^{a(s-1),b|c} \equiv \omega_\mu^{a(s-1),b} h^{c\mu}$*,* $\omega^{a(s-1),bb|c} \equiv \omega_\mu^{a(s-1),bb} h^{c\mu}$*. Contracting (5.64) with* η_{bd} *and, then, symmetrizing* c *with* $a_1 \ldots a_{s-1}$ *results in*

$$\partial_c \omega^{a(s-1),c|a} - \partial^a \omega^{a(s-1),c|}{}_c = 0. \tag{5.65}$$

By symmetrizing in (5.63) $a_1 \ldots a_{s-1}$ *with* d

$$\omega^{a(s-1),c|a} = \partial^c \omega^{a(s-1)|a} - \partial^a \omega^{a(s-1)|c} \tag{5.66}$$

and, then, by contracting (5.63) with $\eta_{da_{s-1}}$

$$\omega^{a(s-1),c|}{}_{c} = (s-1)\left(\partial_c \omega^{a(s-2)c|a} - \partial^a \omega^{a(s-2)c|}{}_{c}\right) \tag{5.67}$$

all the terms of (5.65) can be expressed via $\omega^{a(s-1)|b}$. *Plugging these to (5.65) gives*

$$\Box\omega^{a(s-1)|a} - \partial^a\partial_c\left((s-1)\omega^{a(s-2)c|a} + \omega^{a(s-1)|c}\right) + (s-1)\partial^a\partial^a\omega^{a(s-2)c|}{}_{c} = 0, \tag{5.68}$$

where $\omega^{a(s-1)|a}$ *is to be identified with Fronsdal's field* $\phi^{a(s)}$ *as* $\omega^{a(s-1)|a} = \frac{1}{s}\phi^{a(s)}$, *then,* $\omega^{a(s-2)c|}{}_{c} = \frac{1}{2}\phi^{a(s-2)c}{}_{c}$, $(s-1)\omega^{a(s-2)c|a} + \omega^{a(s-1)|c} = \phi^{a(s-1)c}$ *and the equation acquires the Fronsdal's form (4.13)*

$$\Box\phi^{a(s)} - s\partial^a\partial_c\phi^{a(s-1)c} + \tfrac{s(s-1)}{2}\partial^a\partial^a\phi^{a(s-2)c}{}_{c} = 0. \tag{5.69}$$

For $\mathbf{Y} = \mathbf{Y}\{(s,1)\}$, *i.e.,* $N=1$ *and* $p=1$, *the general scheme of section 3 results in*

$$\begin{array}{rcccccc}
\mathbf{Y}_g : & \begin{array}{|c|}\hline s-1 \\ \hline\end{array} & \dots & \begin{array}{|c|}\hline s-1 \\ \hline \\ \hline\end{array} & \begin{array}{|c|}\hline s \\ \hline \\ \hline\end{array} & \begin{array}{|c|}\hline s+1 \\ \hline s \\ \hline\end{array} & \dots \\
\{n,k\} : & \{1,0\} & \dots & \{1,s-1\} & \{0,0\} & \{0,1\} & \dots \\
g : & g=0 & \dots & g=s-1 & g=s & g=s+1 & \dots \\
q_g : & q_0=1 & \dots & q_{s-1}=1 & q_s=0 & q_{s+1}=0 & \dots ,
\end{array}$$

therefore,

$$\mathcal{W}_{\mathbf{q}} = \begin{cases} \{W_{\mathbf{0}}^{a(s-1)}, W_{\mathbf{0}}^{a(s-1),b}, \dots, W_{\mathbf{0}}^{a(s-1),b(s-1)}\}, & q=0, \\ \{W_{\mathbf{q}}^{a(s-1)}, W_{\mathbf{q}}^{a(s-1),b}, \dots, W_{\mathbf{q}}^{a(s-1),b(s-1)}, W_{\mathbf{q-1}}^{a(s),b(s)}, W_{\mathbf{q-1}}^{a(s+1),b(s)}, \dots\}, & q>0, \end{cases} \tag{5.70}$$

and

$$\sigma_-(W^g_{\mathbf{q}}) = \begin{cases} 0, & g=0, \\ h_c W_{\mathbf{q}}^{a(s-1),b(g-1)c}, & g\in[1,s-1], \\ h_c h_d W_{\mathbf{q-1}}^{a(s-1)c,b(s-1)d}, & g=s, \\ h_c\left(W_{\mathbf{q-1}}^{a(g-s-1)c,b(s)} + \frac{s}{g-s+1}W_{\mathbf{q-1}}^{a(g-s-1)b,b(s-1)c}\right), & g>s. \end{cases} \tag{5.71}$$

With $\mathcal{D} = D_L - \sigma_-$ *full system (5.60) can be rewritten as*

$$\mathcal{D}\omega_{\mathbf{1}} = 0, \qquad \delta\omega_{\mathbf{1}} = \mathcal{D}\xi_{\mathbf{0}}, \tag{5.72}$$

where $\omega_{\mathbf{1}} \in \mathcal{W}_{\mathbf{1}}$ *and* $\xi_{\mathbf{0}} \in \mathcal{W}_{\mathbf{0}}$. *The fields at* $\{n=1, k=0,\dots,s-1\} \sim g \in [0,s-1]$ *form a finite-dimensional* $\mathfrak{iso}(d-1,1)$*-module, whereas the fields with* $\{n=0, k=0,1,\dots\} \sim g \geq s$ *form an infinite-dimensional* $\mathfrak{iso}(d-1,1)$*-module.*

The representatives of cohomology classes can be chosen as $\mathsf{H}^0 = \xi_{\mathbf{0}}^{a(s-1)}$, $\mathsf{H}^1 = h_b\phi^{a(s-1)b}$, $\mathsf{H}^2 = h^b h_c G^{a(s-1)c} - h^a h_c G^{a(s-2)bc} + \gamma\eta^{aa}h^a h_c G^{a(s-4)bcn}{}_{n} + (\beta\eta^{ab}h^a h_c + \alpha\eta^{aa}h^b h_c)G^{a(s-3)cn}{}_{n}$,[15] $\mathsf{H}^3 = h^b h^a h_c \chi^{a(s-2)c}$, $\mathsf{H}^{k>3} = \emptyset$, *where* $\phi^{a(s)}$ *and* $G^{a(s)}$ *are*

[15] $\alpha = -\frac{(s(d+s-5)-d+6)(s-2)}{2(d+s-4)(d+2s-6)}$, $\beta = \frac{(s-2)}{(d+s-4)}$, $\gamma = \frac{(d+s-6)(s-2)(s-3)}{2(d+s-4)(d+2s-6)}$.

double-traceless tensor fields and $\xi^{a(s-1)}$ and $\chi^{a(s-1)}$ are traceless. $\mathsf{H}^{0,1}$ *are nontrivial at the lowest grade, whereas* $\mathsf{H}^{2,3}$ *are nontrivial at grade-one, therefore, the order of equations that are imposed on the* dynamical *field is equal to two. It turns out that there is no need in the explicit form for α, β and γ, it is sufficient to find out the Young symmetry of representatives only. The uniqueness of Fronsdal's theory at the level of action was demonstrated in [67] and at the level of equations in [6, 53]. Consequently, there is no need for equations (5.69) to be found explicitly - those of Fronsdal are the only possible. Again, there is a one-to-one correspondence between the number of fields in* H^1 *and the equations in* H^2*, the gauge parameters in* H^0 *and the Bianchi identities in* H^3*. The system contains an infinitely many of* dual *formulations, those that have field $W_{\mathbf{1}}^{a(s-1),b(k_0)}$ at the lowest grade are gauge* dual *descriptions, whereas those that have field $C^{a(s+k_0),b(s)}$ at the lowest grade are non-gauge.* Dual *description based on field $W_{\mathbf{1}}^{a(s-1),b(1)}$ was elaborated in [68].* ∎

Example 4. ***Unfolding a totally symmetric massless spin-$(s+\frac{1}{2})$ field [69, 70, 31].*** *Unfolding a totally symmetric massless spin-$(s+\frac{1}{2})$ field results in literally the same unfolded system (5.60), (5.72) but with fields taking values in $\mathfrak{so}(d-1,1)$-irreps that are irreducible spin-tensors with the same tensor part as in the bosonic case. Note that operator σ_- is not modified but the σ_- cohomology groups are slightly changed such that the equations become of the first order.*

The Dirac equation. *Unfolded system (5.36) or, equivalently, (5.38) describes a massless spin-$\frac{1}{2}$ field provided that C^k take values in $\boxed{\qquad s \qquad}_{\frac{1}{2}}$ $\mathfrak{so}(d-1,1)$-irreps, i.e., $C^k \equiv C^{\alpha;a(k)}$ and $\Gamma^{\alpha}_{b\,\beta}C^{\beta;ba(k-1)}=0$. Contracting the first equation of (5.36)*

$$\partial_\mu C^\alpha = h_{\mu a}C^{\alpha;a} \tag{5.73}$$

with $h^{b\mu}\Gamma^{\beta}_{b\,\alpha}$ gives the Dirac equation

$$\Gamma^{\alpha}_{a\,\beta}\partial^a C^\beta = 0. \tag{5.74}$$

the rest of equations express auxiliary fields in terms of derivatives of dynamical *field C^α.*

The Rarita-Schwinger equation. *Unfolded system (5.49) or, equivalently, (5.52) describes a massless spin-$\frac{3}{2}$ field provided that irreducible tensors in (5.50) are replaced by the corresponding irreducible tensor-spinors, i.e., $A_{\mathbf{1}} \equiv A^\alpha_\mu$, $C^k \equiv C^{\alpha;[ab],c(k)}$ and $\Gamma^{\alpha}_{b\,\beta}C^{\beta;[ab],c(k)}=0$, $\Gamma^{\alpha}_{d\,\beta}C^{\beta;[ab],dc(k-1)}=0$. Contracting the first equation of (5.49)*

$$\frac{1}{2}(\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu) = h_{\mu a}h_{\nu b}C^{\alpha;[ab]} \tag{5.75}$$

with Γ-matrices and converting world indices μ, ν to the fiber ones according to $A^{\alpha;a} \equiv A^\alpha_\mu h^{a\mu}$ gives the Rarita-Schwinger equation

$$\Gamma^{\alpha}_{b\,\beta}\partial^b A^{\beta;a} - \partial^a\Gamma^{\alpha}_{b\,\beta}A^{\beta;b} = 0, \qquad \delta A^{\alpha;a} = \partial^a\xi^\alpha. \tag{5.76}$$

The rest of unfolded equations express auxiliary fields in terms of derivatives of dynamical *field A^α_μ.*

The Fang-Fronsdal equation for a spin-$(s+\frac{1}{2})$ massless field. *Unfolded system (5.60) or, equivalently, (5.72) describes a massless spin-$(s+\frac{1}{2})$ field provided that irreducible tensors in (5.70) are replaced by the corresponding irreducible tensor-spinors. With all world indices converted to the fiber ones the first equation of (5.60) has the form*

$$G^{\alpha;a(s-1)|[cd]} \equiv \partial^c \omega^{\beta;a(s-1)|d} - \partial^d \omega^{\beta;a(s-1)|c} = \omega^{\beta;a(s-1),c|d} - \omega^{\beta;a(s-1),d|c}, \tag{5.77}$$

where $\omega^{\alpha;a(s-1)|b} \equiv \omega_\mu^{\alpha;a(s-1)} h^{b\mu}$, $\omega^{\alpha;a(s-1),b|c} \equiv \omega_\mu^{\alpha;a(s-1),b} h^{c\mu}$.

Analogously to the bosonic case, nonsymmetric component of $\omega^{\beta;a(s-1)|d}$ *can be gauged away algebraically. Hence, the Fronsdal's field* $\phi^{\alpha;a(s)}$ *is identified as* $\omega^{\alpha;a(s-1)|a} = \frac{1}{s}\phi^{\alpha;a(s)}$. *Then,* $\Gamma^{\alpha}_{b\ \beta}\phi^{\beta;a(s)} = \Gamma^{\alpha}_{b\ \beta}\omega^{\beta;a(s-1)|b}$, $\Gamma^{\alpha}_{b\ \beta}\Gamma^{\beta}_{c\ \gamma}\phi^{\gamma;a(s-2)bc} = \eta_{bc}\omega^{\alpha;a(s-2)b|c}$ *and the Fronsdal's triple* Γ*-traceless constraint is a consequence of irreducibility of* $\omega^{\alpha;a(s-1)|b}$ *in* α, $a_1 \dots a_{s-1}$ *and the lack of any conditions with respect to* $a_1 \dots a_{s-1}$ *and* b. *Since the Fronsdal's field contains three irreducible components with the symmetry of* $\boxed{\quad s \quad}_{\frac{1}{2}}$, $\boxed{\quad s\text{-}1 \quad}_{\frac{1}{2}}$ *and* $\boxed{\quad s\text{-}2 \quad}_{\frac{1}{2}}$, *the equations of motion has to be nontrivial in these three sectors too. The projector on the* dynamical *equations is simply* $\Gamma^{\alpha}_{b\ \beta}G^{\beta;a(s-1),b|a}$ *and gives*

$$\Gamma^{\alpha}_{b\ \beta}\partial^b\phi^{\beta;a(s)} - s\partial^a\Gamma^{\alpha}_{b\ \beta}\phi^{\beta;a(s-1)b} = 0, \tag{5.78}$$

which is in accordance with (4.22). But we would like to note that there are two components with the symmetry of $\boxed{\quad s\text{-}1 \quad}_{\frac{1}{2}}$, *namely,* $\eta_{bc}G^{\alpha;a(s-2)b|ac}$ *and* $\Gamma^{\alpha}_{b\ \beta}\Gamma^{\beta}_{c\ \gamma}G^{\gamma;a(s-1)|[bc]}$. *The correct projector on this component of the equations is given by the combination*

$$2(s-1)\eta_{bc}G^{\alpha;a(s-2)b|ac} - \gamma^{\alpha}_{b\ \beta}\gamma^{\beta}_{c\ \gamma}G^{\gamma;a(s-1)|[bc]}, \tag{5.79}$$

which is identically zero when $G^{\alpha;a(s-1)|[bc]}$ *is expressed via* $\omega^{\beta;a(s-1),c|d}$ *by virtue of (5.77). Therefore, it is (5.79) that does not express certain part of* $\omega^{\beta;a(s-1),c|d}$ *in terms of first derivatives of the dynamical field* $\phi^{\alpha;a(s)}$ *and, thus, this is a dynamical equation. Obviously, (5.79) is a representative of* σ_- *cohomology group* $\mathsf{H}^2_{g=0}$. *In terms of* $\phi^{\alpha;a(s)}$ *the representative has the form*

$$\Gamma^{\alpha}_{b\ \beta}\partial^b\Gamma^{\beta}_{c\ \gamma}\phi^{\gamma;a(s-1)c} - \partial_c\phi^{\alpha;a(s-1)c} + \frac{(s-1)}{2}\partial^a\phi^{\alpha;a(s-2)c}{}_c = 0, \tag{5.80}$$

which is equal to the Γ*-trace of (5.78). The gauge transformations has the form*

$$\delta\phi^{\alpha;a(s)} = \partial^a\xi^{\alpha;a(s-1)}, \tag{5.81}$$

where $\Gamma^{\alpha}_{b\ \beta}\xi^{\beta;a(s-2)b} = 0$.

Consequently, unfolded equations for totally symmetric bosonic fields are proved to completely determine the unfolded equations for totally symmetric fermionic fields. Though σ_- *is not modified in the fermionic case,* σ_- *cohomology groups are changed, since the fermionic* dynamical *equations are of the first order.* ■

6. Unfolding mixed-symmetry fields

First, in section 6.1 an example of the simplest massless mixed-symmetry field of spin-$\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}$ is investigated in detail and leading arguments are given for a spin-$\begin{smallmatrix}\Box\Box\Box\\\Box\end{smallmatrix}$ field. Second, the general statement on arbitrary mixed-symmetry fields is proved, whereas technical details are collected in sections 6.4 and 6.5. The analysis of the number of physical degrees of freedom is carried out in section 6.3.

6.1 Simplest mixed-symmetry fields

Example 5. ***Spin-$\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}$ field.*** *In the* metric-like *approach a spin-$\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}$* field *is minimally described [16, 17] by the subjected to (4.15) field $\phi_{[\mu\mu],\nu}$ that takes values in a reducible $\mathfrak{so}(d-1,1)$-representation $\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}\oplus\Box$, the latter component is identified with the trace $\phi_{\mu\nu,}{}^{\nu}$. The gauge transformations (4.16) has two levels of reducibility, with two gauge parameters $\xi^S_{(\mu\nu)}$, $\xi^A_{[\mu\nu]}$ taking values in $\Box\Box\oplus\bullet$, $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ $\mathfrak{so}(d-1,1)$*-irreps *at the first level and one parameter ξ_μ taking values in $\Box$ $\mathfrak{so}(d-1,1)$*-irrep *at the second level (4.17).*

First, the metric-like *field $\phi_{[\mu\mu],\nu}$ has to be incorporated into certain differential form $e^{\mathbf{Y}}_{\mathbf{q}}$, which is called a* physical *vielbein, the $\mathfrak{so}(d-1,1)$*-irrep *$\mathbf{Y}$ and the degree q to be defined below. To make the symmetries both of the first and of the second level of reducibility manifest the degree of the* physical *vielbein has to be two, at least.*

Moreover, were the physical *vielbein taken to be a degree-one form, not all of the gauge symmetries even at the first level would be manifest. Indeed, there are two possibilities for* physical *vielbein $e^{\mathbf{Y}}_{\mathbf{1}}$ to contain a component with the symmetry of $\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}$. Namely, $\mathbf{Y}=\Box\Box$ and $\mathbf{Y}=\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ since $\Box\Box\otimes\Box=\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}\oplus\Box\Box\Box\oplus\Box$ and $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}\otimes\Box=\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}\oplus\begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}\oplus\Box$. The associated differential gauge parameter is $\xi^{\Box\Box}_{\mathbf{0}}$, in the first case, and $\xi^{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_{\mathbf{0}}$, in the second case. Inasmuch as the gauge parameters are the forms of degree-zero only one of the required two parameters is present in each of the cases.*

The degree of the physical *vielbein has to be not greater than two if the component associated with the* metric-like *field $\phi_{\mu\mu,\nu}$ is required not to be a certain trace part. The reason why a* dynamical *field should not be a certain trace part is explained in the next section.*

Therefore, the physical *vielbein has to be of degree-two and has to take values in $\mathfrak{so}(d-1,1)$*-irrep *that belongs to the vector representation $\Box$, i.e., $e^{\Box}_{\mathbf{2}}\equiv e^a_{\mathbf{2}}\equiv e^a_{\mu\nu}$. The associated gauge parameter $\xi^{\Box}_{\mathbf{1}}\equiv\xi^a_{\mathbf{1}}\equiv\xi^a_\mu$ contains in its Lorentz decomposition $\Box\otimes\Box=\begin{smallmatrix}\Box\\\Box\end{smallmatrix}\oplus\Box\Box\oplus\bullet$ both anti-symmetric $\xi^A_{\mu\nu}=\xi^a_{[\mu}h^b_{\nu]}\eta_{ab}$ and symmetric $\xi^S_{\mu\nu}=\xi^a_{(\mu}h^b_{\nu)}\eta_{ab}$ gauge parameters, the latter enters along with the trace $\xi^a_\mu h^\mu_a$. There exists a second level gauge parameter $\chi^a_{\mathbf{0}}$, which is directly identified with ξ_μ as $\xi_\mu=h^b_\mu\chi^a\eta_{ab}$. There are no* redundant *components in the gauge parameter. However,* physical *vielbein $e^{\Box}_{\mathbf{2}}$ contains one* redundant *component in its decomposition into Lorentz* irreps *$\Box\otimes\begin{smallmatrix}\Box\\\Box\end{smallmatrix}=\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}\oplus\Box\oplus\begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}$. The first and the second components are to be associated with a traceful $\phi_{\mu\mu,\nu}$, whereas the third one, totally anti-symmetric component $e^{[a|bc]}$ of $e^{a|bc}=e^a_{\mu\nu}h^{b\mu}h^{c\nu}$, is a* redundant *one and has to be made non-dynamical. In order to do so, algebraic gauge parameter $\xi^{[abc]}_{\mathbf{0}}$*

with the symmetry of $\yng(1,1,1)$ *is introduced. It is obvious from pure algebraic gauge law*

$$\delta e^a_{\mathbf{2}} = h_b h_c \xi^{[abc]}_{\mathbf{0}} \tag{6.1}$$

that the redundant *component can be fixed to zero and* $e^a_{\mathbf{2}}$ *can be directly identified with* $\phi_{\mu\mu,\nu}$ *as* $e^a_{\mu\mu} = \phi_{\mu\mu,\nu} h^{a\nu}$. *From gauge transformation laws* $\delta e^a_{\mathbf{2}} = d\xi^a_{\mathbf{1}}$, $\delta \xi^a_{\mathbf{1}} = d\chi^a_{\mathbf{0}}$, *i.e.,* $\delta e^a_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \xi^a_\nu - \partial_\nu \xi^a_\mu\right)$, $\delta \xi^a_\mu = \partial_\mu \chi^a$, *required gauge transformations (4.16), (4.17) for* metric-like *field* $\phi_{\mu\mu,\nu}$ *and for the first order gauge parameters* $\xi^S_{(\mu\nu)}$, $\xi^A_{[\mu\nu]}$ *are easily recovered.*

Since in the unfolded approach each gauge parameter possesses its own gauge field and vice-verse, associated with $\xi^{[abc]}_{\mathbf{0}}$ *gauge field* $\omega^{[abc]}_{\mathbf{1}}$ *enters as*

$$de^a_{\mathbf{2}} = h_b h_c \omega^{[abc]}_{\mathbf{1}}, \tag{6.2}$$

which determines the first equation. Applying d to this equation results in Bianchi identity $h_b h_c d\omega^{[abc]}_{\mathbf{1}} \equiv 0$, *which has a unique solution of the form*

$$d\omega^{[abc]}_{\mathbf{1}} = h_d h_f C^{[abc],[df]}_{\mathbf{0}}, \tag{6.3}$$

with $C^{[abc],[df]}_{\mathbf{0}}$ *having the symmetry* $\yng(2,2,1)$ *of the generalized Weyl tensor of a spin-*$\yng(2,1)$ field, *i.e., it is anti-symmetric in* $[abc]$, $[df]$ *and* $C^{[abc,d]f}_{\mathbf{0}} \equiv 0$. *To clarify the form of the solution let* $C^{[abc]|[df]}_{\mathbf{0}}$ *be a degree-zero form anti-symmetric in* $[abc]$, $[df]$ *and with no definite symmetry between these two groups of indices. The Bianchi identity implies* $h_b h_c h_d h_f C^{[abc]|[df]}_{\mathbf{0}} \equiv 0$, *which is equivalent to* $C^{a[bc|df]}_{\mathbf{0}} \equiv 0$ *since vielbeins* h^a *anticommute. Therefore, the solution is parameterized by those components of* $C^{[abc]|[df]}_{\mathbf{0}} \sim \yng(1,1,1) \otimes \yng(1,1)$ *that has the symmetry of Young diagrams with no more than three rows, since the requirement for total anti-symmetrization of any four indices to give zero is the characteristic property of Young diagrams with at most three rows.* $\yng(1,1,1) \otimes \yng(1,1) \ni \yng(2,2,1)$ *is the only component of zero trace order*[16] *with three rows and there are no components with the number of rows less than three. Alternatively, one could search for the solution in the sector of degree-one forms as* $d\omega^{[abc]}_{\mathbf{1}} = h_d C^{[abc]|d}_{\mathbf{1}}$, *with some* $C^{[abc]|d}_{\mathbf{1}}$, *but Bianchi identity* $h_b h_c h_d C^{[abc]|d}_{\mathbf{1}} \equiv 0$ *implies that the only component of* $C^{[abc]|d}_{\mathbf{1}} \sim \yng(1,1,1) \otimes \yng(1)$ *that is allowed must have less than three rows, which is impossible since* $[abc]$ *makes Young diagrams of* $\yng(1,1,1) \otimes \yng(1)$ *consist of three rows, at least.*[17] *Thus, one has to search for the solution among the forms of less degree.*

---

[16] Although, there are trace components with the number of rows less than four, e.g., $\yng(1,1,1)$, their tensor product by the metric η^{ab} contains components with the symmetry of the $sl(d)$-Young diagrams with more than three rows. Nontrivial trace with the symmetry of $\yng(2,1)$ corresponds to the mass-like term. Hence, traceful tensors either do not satisfy the Bianchi identities or introduce mass-like terms.

[17] Again, the trace component $\yng(1,1)$ enters the tensor product as $\eta \otimes \yng(1,1)$ and has the number of rows not less than three.

Further unfolding requires a set of fields $C^{[abc],[df],g(k)}$ with the symmetry of *to be introduced and the full system has the form*

$$\begin{aligned}
de^a_{\mathbf{2}} &= h_b h_c \omega^{[abc]}_{\mathbf{1}}, & \delta e^a_{\mathbf{2}} &= d\xi^a_{\mathbf{1}} + h_b h_c \xi^{[abc]}_{\mathbf{0}}, \qquad \delta\xi^a_{\mathbf{1}} = d\chi^a_{\mathbf{0}},\\
d\omega^{[abc]}_{\mathbf{1}} &= h_d h_f C^{[abc],[df]}_{\mathbf{0}}, & \delta\omega^{[abc]}_{\mathbf{1}} &= d\xi^{[abc]}_{\mathbf{0}},\\
dC^{[abc],[df],g(k)}_{\mathbf{0}} &= h_v C^{[abc],[df],g(k)v}_{\mathbf{0}}.
\end{aligned} \tag{6.4}$$

Consequently, the unfolded system incorporates field $\phi_{\mu\mu,\nu}$ with all required differential gauge parameters at all levels of reducibility; the redundant *component of the* physical *vielbein does not contribute to the dynamics. $\omega^{[abc]}_{\mathbf{1}} \sim \otimes = \oplus \oplus$, the first component is a field strength for the* redundant *field, once the field has been gauged away, the associated field strength is zero and $\omega^{[abc]}_{\rho} = h^{a\mu}h^{b\nu}h^{c\lambda}T_{[\mu\nu\lambda],\rho}$ for some $T_{[\mu\nu\lambda],\rho}$, $T_{[\mu\nu\lambda,\rho]} \equiv 0$, which incorporates both and (as the trace). To recover equations (4.15) in terms of* metric-like *fields it is convenient to convert all fiber indices to the world ones in the first two equations*

$$\partial_{[\mu}e^a_{\nu\lambda]} = h_{b[\mu}h_{c\nu}\omega^{abc}_{\lambda]}, \tag{6.5}$$

$$\partial_{[\mu}\omega^{abc}_{\nu]} = h_{d[\mu}h_{f\nu]}C^{abc,df}, \tag{6.6}$$

to substitute $T_{\mu\nu\lambda,\rho}$ and to contract two indices in the second equation

$$\partial_{[\mu}\phi_{\nu\lambda],\rho} = T_{\mu\nu\lambda,\rho}, \tag{6.7}$$

$$\partial^\rho T_{\mu\nu\rho,\lambda} - \partial_\lambda T_{\mu\nu\rho,}{}^{\rho} = 0. \tag{6.8}$$

Plugging (6.7) in (6.8) gives equation (4.15).

For $\mathbf{Y} = \mathbf{Y}\{(2,1),(1,1)\}$, i.e., $N = 2$ and $p = 2$, the general scheme of section 3 results in

$$\begin{array}{rccccc}
\mathbf{Y}_g: & & & & & \dots\\
\{n,k\}: & \{2,0\} & \{1,0\} & \{0,0\} & \{0,1\} & \dots\\
g: & g=0 & g=1 & g=2 & g=3 & \dots\\
q_g: & q_0=2 & q_1=1 & q_2=0 & q_3=0 & \dots,
\end{array}$$

therefore,

$$\mathcal{W}_{\mathbf{q}} = \begin{cases}\{W^a_{\mathbf{0}}\}, & q=0,\\ \{W^a_{\mathbf{1}}, W^{[abc]}_{\mathbf{0}}\}, & q=1,\\ \{W^a_{\mathbf{q}}, W^{[abc]}_{\mathbf{q-1}}, W^{[abc],[df]}_{\mathbf{q-2}}, W^{[abc],[df],g}_{\mathbf{q-2}}, \dots\}, & q>1,\end{cases} \tag{6.9}$$

and

$$\sigma_-(W^g_{\mathbf{q}}) = \begin{cases}0, & g=0,\\ h_b h_c W^{[abc]}_{\mathbf{q-1}}, & g=1,\\ h_b h_c W^{[aaa],[bc]}_{\mathbf{q-2}}, & g=2,\\ h_d W^{[aaa],[bb],c(g-3)d}_{\mathbf{q-2}}, & g>2.\end{cases} \tag{6.10}$$

with $\mathcal{D} = D_L - \sigma_-$ *unfolded system (6.4) can be rewritten as*

$$\mathcal{D}\omega_{\mathbf{2}} = 0, \qquad \delta\omega_{\mathbf{2}} = \mathcal{D}\xi_{\mathbf{1}}, \qquad \delta\xi_{\mathbf{1}} = \mathcal{D}\xi_{\mathbf{0}} \tag{6.11}$$

where $\omega_{\mathbf{2}} \in \mathcal{W}_{\mathbf{2}}$, $\xi_{\mathbf{1}} \in \mathcal{W}_{\mathbf{1}}$ *and* $\xi_{\mathbf{0}} \in \mathcal{W}_{\mathbf{0}}$. *The fields at* $\{n = 2, k = 0\} \sim g = 0$ *and* $\{n = 1, k = 0\} \sim g = 1$ *form two finite-dimensional* $\mathfrak{iso}(d-1,1)$*-modules, whereas the fields with* $\{n = 0, k = 0, 1, \ldots\} \sim g \geq 2$ *form an infinite-dimensional* $\mathfrak{iso}(d-1,1)$*-module.* ■

Example 6. ***Spin-*$\yng(3,2)$ *field (briefly).*** *In the case of a massless spin-*$\yng(3,2)$ field, *there are two gauge parameters at the first level of reducibility, which have the symmetry*[18] *of* $\yng(2,2)$, $\yng(3,1)$, *and one gauge parameter at the second level with the symmetry of* $\yng(2,1)$. *First, for* $\mathbf{Y} = \mathbf{Y}\{(3,1),(2,1)\}$ *the proposed scheme results in*

$$\begin{array}{rcccccc}
\mathbf{Y}_g: & \yng(2,1) & \yng(2,1,1) & \yng(2,2,2) & \yng(3,3,2) & \yng(4,3,2) & \ldots \\
\{n,k\}: & \{2,0\} & \{2,1\} & \{1,0\} & \{0,0\} & \{0,1\} & \ldots \\
g: & g=0 & g=1 & g=2 & g=3 & g=4 & \ldots \\
q_g: & q_0 = 2 & q_1 = 2 & q_2 = 1 & q_3 = 0 & q_4 = 0 & \ldots
\end{array}$$

therefore, the physical *vielbein has to be a two-form taking values in* $\yng(2,1)$, *i.e.,* $e_{\mathbf{2}}^{\yng(2,1)} \equiv e_{\mathbf{2}}^{aa,b}$. *Associated gauge parameter* $\xi_{\mathbf{1}}^{\yng(2,1)} \equiv \xi_{\mathbf{1}}^{aa,b}$ *has components* $\yng(2,1) \otimes \yng(1) = \yng(3,1) \oplus \yng(2,2) \oplus \yng(2,1,1)$, *the first two having the symmetry of the required gauge parameters, with the third one being* redundant. *The last three components in the decomposition of the* physical *vielbein* $\yng(2,1) \otimes \yng(1,1) = \yng(3,2) \oplus \yng(3,1,1) \oplus \yng(2,2,1) \oplus \yng(2,1,1,1)$ *are also* redundant. *By virtue of* Stueckelberg *symmetry with parameter* $\xi_{\mathbf{1}}^{\yng(2,1,1)}$ *these three components can be gauged away, inasmuch as* $\yng(2,1,1) \otimes \yng(1) = \yng(3,1,1) \oplus \yng(2,2,1) \oplus \yng(2,1,1,1)$. *The* redundant *component* $\yng(2,1,1)$ *of the first level gauge parameter* $\xi_{\mathbf{1}}^{\yng(2,1)}$ *also can be gauged away by virtue of* Stueckelberg *symmetry with level-two gauge parameter* $\xi_{\mathbf{0}}^{\yng(2,1,1)}$. *Consequently, at least the fact that the unfolded system incorporates the* metric-like *field, all the required gauge parameters, and* redundant *components do not contribute to the dynamics is proved.*■

6.2 General mixed-symmetry fields

Given a massless field of spin $\mathbf{Y} = \mathbf{Y}\{(s_1,p_1),\ldots,(s_N,p_N)\}$, *described by a* metric-like *field* $\phi_{\mathbf{Y}}(x)$, *taking values in the* irrep *of the Lorentz algebra* $\mathfrak{so}(d-1,1)$ *with the same symmetry type(*minimal formulation*) there exists a unique unfolded system with the* physical *vielbein at the* lowest grade *and all symmetries being manifest that reproduces the original* metric-like *system. The system has the form (3.1).*

Sketch of the proof 1.

[18] To simplify the example no consideration is given to the trace components of the fields.

1. *The lowest grade field (*physical *vielbein)* $e^{\mathbf{Y}_0}_{\mathbf{q_0}}$ *turns out to be completely determined by the requirement for it to contain the* metric-like *field and for its gauge parameters* $\xi^{\mathbf{Y}_0}_{\mathbf{q_0-k}}$, $k>0$ *to contain all the necessary gauge parameters at all levels of reducibility (4.19).*

2. $e^{\mathbf{Y}_0}_{\mathbf{q_0}}$ *and its gauge parameters appear to contain* redundant *components that have to be made non-dynamical. The only way to achieve this is to introduce* Stueckelberg *symmetry* $\delta e^{\mathbf{Y}_0}_{\mathbf{q_0}} = \sigma^1_-(\xi^{\mathbf{Y}_1}_{\mathbf{q_1-1}})$, $\delta\xi^{\mathbf{Y}_0}_{\mathbf{q_0-1}} = \sigma^1_-(\xi^{\mathbf{Y}_1}_{\mathbf{q_1-2}})$, *...with certain* $\xi^{\mathbf{Y}_1}_{\mathbf{q_1-k}}$, $k>0$, *which turns out to be unambiguously defined by this requirement.*

3. $\xi^{\mathbf{Y}_1}_{\mathbf{q_1-1}}$ *has associated gauge field* $\omega^{\mathbf{Y}_1}_{\mathbf{q_1}}$ *and gauge transformations* $\delta\omega^{\mathbf{Y}_0}_{\mathbf{q_0}} = \sigma^1_-(\xi^{\mathbf{Y}_1}_{\mathbf{q_1-1}})$ *completely determines the first equation* $de^{\mathbf{Y}_0}_{\mathbf{q_0}} = \sigma^1_-(\omega^{\mathbf{Y}_1}_{\mathbf{q_1}})$, *which implies Jacobi identity* $\sigma^1_-(d\omega^{\mathbf{Y}_1}_{\mathbf{q_1}}) \equiv 0$. *The general solution is proved in section 6.4 to have the form* $d\omega^{\mathbf{Y}_1}_{\mathbf{q_1}} = \sigma^2_-(\omega^{\mathbf{Y}_2}_{\mathbf{q_2}})$ *with certain* $\omega^{\mathbf{Y}_2}_{\mathbf{q_2}}$. *The second equation implies the second Jacobi identity* $\sigma^2_-(d\omega^{\mathbf{Y}_2}_{\mathbf{q_2}}) \equiv 0$ *and so on.*

4. *Once, the total unfolded system is defined, the proof of the facts that (i) the correct* dynamical *equations are imposed; (ii) there are no other* dynamical *fields or other differential gauge symmetries in the system; is obtained through the calculations of* σ_- *cohomology groups. Ignoring the trace pattern of fields, i.e., for traceful tensors, the proof of (ii) can be simplified and is done in this section. Even stronger statement that there exists a duality* $\mathsf{H}^{p+1+k} \sim \mathsf{H}^{p-k}$ *is proved in section 6.5.*

First, as it is clear from the examples above, the *metric-like* field $\phi_{\mathbf{Y}_M}(x)$ has to be incorporated into a differential form $e^{\mathbf{Y}_0}_{\mathbf{q}}$ of degree-q taking values in certain $\mathfrak{so}(d-1,1)$-*irrep* $\mathbf{Y}_0$ (for this reason $e^{\mathbf{Y}_0}_{\mathbf{q}}$ is called *physical* vielbein). Having converted all world indices to fiber ones in accordance with (5.29), $\mathfrak{so}(d-1,1)$-tensor product $\mathbf{Y}_0 \otimes \mathbf{Y}\{(1,q)\}$ of $\mathbf{Y}_0$ with one column diagram of the height q must contain the component with the symmetry of $\mathbf{Y}_M$. In the case of the minimal formulation $\mathbf{Y}_M = \mathbf{Y}$, where $\mathbf{Y} = \mathbf{Y}\{(s_1,p_1),\dots,(s_N,p_N)\}$ characterizes the spin.

It is useful to introduce a notion of the quotient of two Young diagrams: let the quotient of two Young diagrams $\mathbf{Y}_1/\mathbf{Y}_2$ be a direct sum of those diagrams, whose $\mathfrak{so}(d-1,1)$-tensor product by $\mathbf{Y}_2$ contains $\mathbf{Y}_1$. For the first sight, given $q \geq 0$, any element Q of $\mathbf{Y}_M/\mathbf{Y}\{(1,q)\}$ might be chosen as $\mathbf{Y}_0$. If q is greater than the height of $\mathbf{Y}_M$ the component with the symmetry $\mathbf{Y}_M$ in the tensor product $Q \otimes \mathbf{Y}\{(1,q)\}$ has to be certain trace, inevitably.

Although, there is no apparent problems in considering unfolded systems with dynamical field hidden in certain trace component of the *physical* vielbein, e.g., Maxwell gauge potential A_μ might be identified with the trace $e^a_{\mu\nu}h^\mu_a$ of two-form $e^a_{\mu\nu}$. However, this exotic incorporating of the *physical* field does not take place in the already known systems. Moreover, it can be proved that such exotic systems do not exist, if irreducible. Nevertheless, they might be a part of some reducible system, which describes more than one *field*. Therefore, we require the degree q of the *physical* vielbein to be not greater than the height of $\mathbf{Y}_M$.

Inasmuch as

1. the equations imposed on $\phi_{\mathbf{Y}}(x)$ possess reducible gauge transformations with the number of levels equal to the height $p=\sum_{i=1}^{i=N} p_i$ of $\mathbf{Y}$;

2. for a degree-q gauge field of an unfolded system there exist q levels of gauge transformations;

the degree q of *physical* vielbein $e^{\mathbf{Y}_0}_{\mathbf{q}}$ must be equal to p.

The quotient $\mathbf{Y}/\mathbf{Y}\{(1,p)\}$ contains only one element (provided $\phi_{\mathbf{Y}}(x)$ is forbidden to be a trace component), it is the diagram (3.2) with the symmetry of $\mathbf{Y}_0 = \mathbf{Y}\{(s_1-1,p_1),\dots,(s_N-1,p_N)\}$, i.e., it is equal to $\mathbf{Y}$ without the first column. Therefore, *physical* vielbein $e^{\mathbf{Y}_0}_{\mathbf{p}}$ is completely defined. So does gauge parameters $\xi^{\mathbf{Y}_0}_{\mathbf{p-k}}$ at the k-th level of reducibility, $k\in[1,p]$. The requirement for tensor product $\mathbf{Y}_0\otimes\mathbf{Y}\{(1,p-k)\}$ to contain all gauge parameters ξ^{i_k} given by (4.19) at the k-th level of reducibility is also satisfied. Consequently, the whole pattern of fields/gauge parameters of the *metric-like* formulation is reproduced. Note that $\mathbf{Y}_0\equiv\mathbf{Y}_0$ and $q_0=p$ in the terms of section 3.

Let us note that if one writes down the *physical* vielbein explicitly as $e^{a_1(s_1-1),\dots,a_p(s_p-1)}_{\mu_1\dots\mu_p}$, where s_i are the lengths of the rows of $\mathbf{Y}$ and converts the form indices to the tangent ones by virtue of inverse background vielbein $h^{\mu a}$

$$e^{a_1(s_1-1),\dots,a_p(s_p-1)|[d_1\dots d_p]}=e^{a_1(s_1-1),\dots,a_p(s_p-1)}_{\mu_1\dots\mu_p}h^{\mu_1 d_1}\dots h^{\mu_p d_p}, \tag{6.12}$$

dynamical field $\phi_{\mathbf{Y}}(x)$ should be identified with $e^{a_1(s_1-1),\dots,a_p(s_p-1)|a_1\dots a_p}$, which has the symmetry of $\mathbf{Y}$ with some traces included. Consequently, the generalized Labastida's double-tracelessness condition (4.20) [20] is obvious, inasmuch as the contraction of two metric tensor $\eta^{a_i a_i}\eta^{a_i a_i}$ with the i-th group of indices vanishes.

Generally, in addition to $\phi_{\mathbf{Y}}(x)$, decomposition $\mathbf{Y}_0\otimes\mathbf{Y}\{(1,p)\}$ of *physical* vielbein $e^{\mathbf{Y}_0}_{\mathbf{p}}$ into Lorentz *irreps* contains *redundant* components, which must not contribute to the physical degrees of freedom. So does $\mathbf{Y}_0\otimes\mathbf{Y}\{(1,p-k)\}$, i.e., in addition to ξ^{i_k} it contains a lot of components that can not be made genuine differential gauge parameters. There are only two possible ways to get rid of *redundant* fields in the unfolded formalism:

A. *redundant* components could be directly fixed to zero by algebraic (*Stueckelberg*) symmetry.

B. *redundant* components might be auxiliary fields for other fields (at lower grade) and so on. The process stops since the grade is assumed to be bounded from below.

We require *dynamical* fields incorporated in the *physical* vielbein to be at the lowest grade. Actually, (B) takes place for *dual* gauge descriptions, but not for the minimal. Therefore, the only possibility to make *redundant* components to be non-dynamical is to introduce a *Stueckelberg* symmetry with certain parameters $\xi^{\mathbf{Y}_1}_{\mathbf{q_1-k}}$

$$\delta e^{\mathbf{Y}_0}_{\mathbf{p}}=d\xi^{\mathbf{Y}_0}_{\mathbf{p-1}}+\sigma^1_-(\xi^{\mathbf{Y}_1}_{\mathbf{q_1-1}}), \tag{6.13}$$

$$\delta\xi^{\mathbf{Y}_0}_{\mathbf{p-1}}=d\xi^{\mathbf{Y}_0}_{\mathbf{p-2}}+\sigma^1_-(\xi^{\mathbf{Y}_1}_{\mathbf{q_1-2}}), \tag{6.14}$$

$$\dots\dots, \tag{6.15}$$

where σ_-^1 is a certain operator built of background vielbein h^a that contracts a number of indices of $\mathbf{Y}_1$ with $h^a \dots h^c$ to obtain $\mathbf{Y}_0$. Diagram[19] $\mathbf{Y}_1$ and form degree q_1 can be easily found since $\mathbf{Y}_1 \otimes \mathbf{Y}\{1, q_1 - i\}$, $i > 0$ must contain all *redundant* components of $e_{\mathbf{p}}^{\mathbf{Y}_0}$ and $\xi_{\mathbf{p-k}}^{\mathbf{Y}_0}$.

In the unfolded formalism each gauge field possesses its own gauge parameter and vice-versa. Therefore, there is a gauge field $\omega_{\mathbf{q_1}}^{\mathbf{Y}_1}$, which contributes to the equations for $e_{\mathbf{p}}^{\mathbf{Y}_0}$ as

$$de_{\mathbf{p}}^{\mathbf{Y}_0} = \sigma_-^1 \left(\omega_{\mathbf{q_1}}^{\mathbf{Y}_1}\right). \tag{6.16}$$

The first equation determines the first Jacobi identity of the form $\sigma_-^1(d\omega_{\mathbf{q_1}}^{\mathbf{Y}_1}) \equiv 0$, which can be solved as

$$d\omega_{\mathbf{q_1}}^{\mathbf{Y}_1} = \sigma_-^2(\omega_{\mathbf{q_2}}^{\mathbf{Y}_2}) \tag{6.17}$$

for certain gauge field $\omega_{\mathbf{q_2}}^{\mathbf{Y}_2}$ and operator σ_-^2 that satisfies $\sigma_-^1\sigma_-^2 \equiv 0$. The general solution of Jacobi identities is given in section 6.4. From now on let the superscripts of σ_- be omitted. The second equation determines the second Jacobi identity $\sigma_-(d\omega_{\mathbf{q_2}}^{\mathbf{Y}_2}) \equiv 0$, which can be also solved and so on. Consequently, the knowledge of the lowest grade gauge field $e_{\mathbf{p}}^{\mathbf{Y}_0}$ and of the *Stueckelberg* symmetries required to get rid of *redundant* components determines the first equation, which by virtue of Jacobi identities determines the second and so on. The total unfolded system has the structure

$$\begin{aligned}
d\omega_{q_g}^{\mathbf{Y}_g} &= \sigma_-\left(\omega_{q_{g+1}}^{\mathbf{Y}_{g+1}}\right), \quad g = 0, 1, \dots, \\
\delta\omega_{q_g}^{\mathbf{Y}_g} &= d\xi_{q_g}^{\mathbf{Y}_g} + \sigma_-\left(\xi_{q_{g+1}-1}^{\mathbf{Y}_{g+1}}\right), \\
\delta\xi_{q_g}^{\mathbf{Y}_g} &= \cdots .,
\end{aligned} \tag{6.18}$$

where σ_- is certain algebraic operator built of background vielbein h^a and $(\sigma_-)^2 = 0$.

The only facts remain to be proved are that there are no other *dynamical* fields and that the proper correspondence (duality) between the cohomology groups holds true. To this end it is sufficient to analyze only the $\mathfrak{so}(d-1,1)$-*irreps* content of H_g^q, i.e., only the symmetry type and the multiplicity of irreducible Lorentz tensors that by virtue of (5.30) are the representatives of H_g^q.

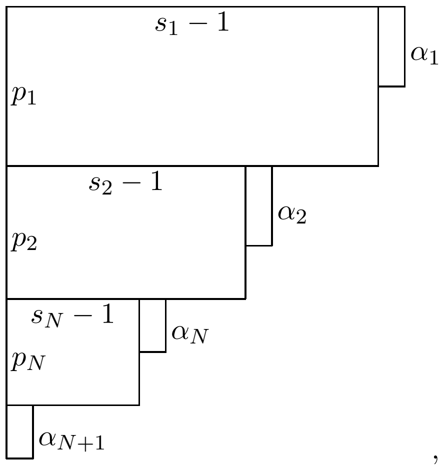
It is proved in the next section that there is a duality $\mathsf{H}_{g=0}^{p-k} \sim \mathsf{H}_{g=1}^{p+k+1}$ between the σ_- cohomology groups, which reveals the one-to-one correspondence between the gauge *dynamical* fields and equations of motion, the level-k gauge symmetries and the level-k Bianchi identities. The representatives of cohomology groups in the sector of fields and gauge symmetries directly correspond to those of *metric-like* approach and there is no other nontrivial cohomology in these sectors.

Though, the trace pattern of fields/gauge parameters is also important and the calculation of σ_- cohomology groups provides a comprehensive answer to this question, the very procedure being a bit technical. Ignoring the trace pattern of the fields, it can be easily proved that the required *metric-like* field is incorporated in *physical* vielbein $e_{q_0}^{\mathbf{Y}_0}$,

[19] The specific character of Minkowski massless fields is in that all *Stueckelberg* parameters can be incorporated into a single $\mathbf{Y}_1$-valued form, this not being the case for (anti)-de Sitter massless fields.

all the *redundant* components of the *physical* vielbein can be gauged away by a pure algebraic symmetry and the same holds for differential gauge parameters $\xi^{\mathbf{Y}_0}_{q_0-k}$ at all levels of reducibility, i.e., the required pattern of gauge parameters is recovered, whereas all the *redundant* components are either *Stueckelberg* or *auxiliary*. This is the necessary condition only and the traces has to be taken into account. Also, it is still to be proved that the correct equations of motions are imposed. Provided that there are no trace conditions on the fields, the system is referred to as *off-shell* [65]. An *off-shell* system may contain only constraints that express *auxiliary* fields via the derivatives of the *dynamical* field, imposing no restrictions on the latter.

The off-shell system. Technically, ignoring the traces is equivalent to the replacement of all $\mathfrak{so}(d-1,1)$-*irreps* by the $sl(d)$-*irreps* that are characterized by the same Young diagrams, i.e., instead of taking $\mathfrak{so}(d-1,1)$-tensor products one needs to apply the $sl(d)$-tensor product's rules, which are much simpler. The decomposition of the *physical* vielbein $e^{\mathbf{Y}_0}_{\mathbf{q_0}}$ into $sl(d)$-*irreps* is given by diagrams $\mathbf{Y}_{\{\alpha_i\}}$ of the form



, (6.19)

where $\alpha_1+\cdots+\alpha_{N+1}=q_0=p$. The decomposition of *Stueckelberg* gauge parameter $\xi^{\mathbf{Y}_1}_{\mathbf{q_1-1}}$ for $e^{\mathbf{Y}_0}_{\mathbf{q_0}}$ has the components of the same form but with $\alpha_{N+1}\geq 1$ because $\mathbf{Y}_1$ (3.3) has already the form of $\mathbf{Y}_0$ with one cell in the bottom-left. Therefore, all components of $e^{\mathbf{Y}_0}_{\mathbf{q_0}}$ except for those with $\alpha_{N+1}=0$ are of *Stueckelberg* type, but there is only one component of $e^{\mathbf{Y}_0}_{\mathbf{q_0}}$ with $\alpha_{N+1}=0$, namely, it has $\alpha_i=p_i$, $i\in[1,N]$ and, hence, has the symmetry of $\mathbf{Y}$. The decomposition of level-k differential gauge parameter $\xi^{\mathbf{Y}_0}_{\mathbf{q_0-k}}$ has the form (6.19) with $\alpha_1+\cdots+\alpha_{N+1}=p-k$ and, again, all the components of $\xi^{\mathbf{Y}_0}_{\mathbf{q_0-k}}$ with $\alpha_{N+1}\geq 1$ can be gauged away by pure algebraic symmetry with $\xi^{\mathbf{Y}_1}_{\mathbf{q_1-k-1}}$. Consequently, there is a complete matching between (4.19) and those in $\xi^{\mathbf{Y}_0}_{\mathbf{q_0-k}}$ that are not pure gauge themselves. It can be easily proved also that, if ignoring the traces, there are no other *dynamical* fields/differential gauge parameters at higher grade. Consequently,

Lemma 1. $\mathsf{H}(\sigma_-)$ *with respect to* $sl(d)$ *are given by*

$$\mathsf{H}^k(\sigma_-)_{sl(d)}=\begin{cases}\mathbf{Y}_{\{\alpha_i\}}: g=0, \quad \alpha_1+\cdots+\alpha_N=k, \quad \alpha_{N+1}=0, & k\leq p\\ \emptyset, & k>p.\end{cases} \tag{6.20}$$

The triviality of the higher $k > p$ cohomology groups for $sl(d)$, which should contain equations of motion and Bianchi identities, is expected since the system is *off-shell.* For the equations to have the second order in derivatives the only nontrivial cohomology group H^{p+1} must be at the grade-one, i.e., $\mathsf{H}^{p+1}_{g=1} \neq \emptyset$. As is seen from the examples above, the equations correspond to those representatives of $\mathsf{H}^{p+1}_{g=1}$ that are certain traces. Indeed, the total rank $|W^{\mathbf{Y}_g}_{\mathbf{q_g}}|$, which is equal to the sum $q_g + |\mathbf{Y}_g|$ of degree q_g and rank of $\mathbf{Y}_g$, is preserved by σ_-. Therefore, $|W^{\mathbf{Y}_g}_{\mathbf{q_g}}| = |W^{\mathbf{Y}_0}_{\mathbf{q_0}}| + g$ and $|R^{\mathbf{Y}_1}_{\mathbf{q_1+1}}| = |\omega^{\mathbf{Y}_0}_{\mathbf{q_0}}| + 2 = |\mathbf{Y}| + 2$, where $e^{\mathbf{Y}_0}_{\mathbf{q_0}} \in \mathcal{W}^{g=0}_{\mathbf{p}}$ and $R^{\mathbf{Y}_1}_{\mathbf{q_1+1}} \in \mathcal{W}^{g=1}_{\mathbf{p+1}}$ contain *metric-like* field $\phi_{\mathbf{Y}_M}(x)$ and the equations on $\phi_{\mathbf{Y}}(x)$, respectively. Consequently, the representative of $\mathsf{H}^{p+1}_{g=1}$ that corresponds to the equations on the traceless part of $\phi_{\mathbf{Y}}(x)$ has to be identified with certain trace of the first order and so on for the traces of $\phi_{\mathbf{Y}}(x)$.

The calculation of σ_- cohomology groups carried out in section 6.5 implies

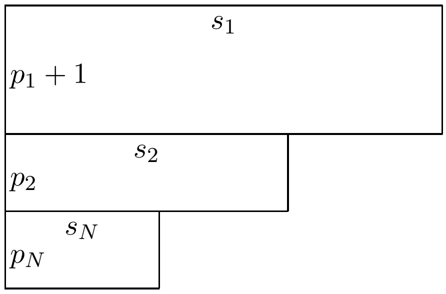
1. The only nontrivial cohomology groups are $\mathsf{H}^{p-k}_{g=0}$ and $\mathsf{H}^{p+k+1}_{g=1}$, $k = 1, \dots, p$. Therefore, the *dynamical* fields and independent gauge parameters belong to the zero grade $\mathcal{W}^{g=0}_{\mathbf{q}}$ subspaces; the equations are of the second order and the Bianchi identities for the k-th level gauge symmetries are of the $(k+2)$-th order.

2. $\mathsf{H}^{p-k}_{r,g=0} \sim \mathsf{H}^{p+k+1}_{r+k+1,g=1}$, i.e., there is a one-to-one correspondence between the elements of $\mathsf{H}^{p-k}_{g=0}$ that are traces of the r-th order and the elements of $\mathsf{H}^{p+k+1}_{g=1}$ that are traces of the $(r + k + 1)$-th order. Roughly speaking, $\mathsf{H}^{p} \sim \mathsf{H}^{p+1}$, $\mathsf{H}^{p-1} \sim \mathsf{H}^{p+2}$ and so on. Therefore, there is a one-to-one correspondence between the *dynamical* fields and the equations of motion, the level-k gauge symmetries and the k-th Bianchi identities.

3. The $\mathfrak{so}(d-1,1)$-*irreps* that correspond to the elements of $\mathsf{H}^{p-k}(\sigma_-)$ and are the traces of the zeroth order are given by the $\mathfrak{so}(d-1,1)$-Young diagrams that have the form (6.19) with $\alpha_{N+1} = 0$ and $\alpha_1 + \dots + \alpha_N = p - k$, which exactly reproduce the required pattern (4.19). Note that certain higher order traces are also the elements of cohomology groups. These fields represent the 'auxiliary' fields of the *metric-like* formulation, which had to be introduced to make the gauge symmetry *off-shell.* To prove the theorem it is not necessary to know the concrete trace pattern, the duality between the cohomology groups is sufficient. The details of the trace pattern are in section 6.5.

In the fermionic case the traces are substituted for Γ-traces. Important is that acting on tensor indices only, σ_- does not break down the irreducibility of spin-tensors. The duality has the form $\mathsf{H}^{p-k}_{r,g=0} \sim \mathsf{H}^{p+k+1}_{r+k+1,g=0}$ or simply $\mathsf{H}^{p-k}_{g=0} \sim \mathsf{H}^{p+k+1}_{g=0}$, which means that equations are of the first order and the duality between fields/equations, gauge symmetries/Bianchi identities takes place, which completes the proof.

Note, that $\mathbf{Y}_{g=s_1} \equiv \mathbf{Y}_{\{n=0,k=0\}}$ has the symmetry of the generalized Weyl tensor for a spin-$\mathbf{Y}$ massless *field*



$$. \tag{6.21}$$

Analogously to the examples of section 5.3, massless mixed-symmetry fields can be described by the same unfolded system (3.6) that is restricted to $\mathcal{W}^g_{\mathbf{q}}$ with $g \geq g_0$, i.e., the system contains infinitely many *dual* formulations. As the generalized Weyl tensor and its descendants are non-gauge fields, the *dual* descriptions with $g_0 \geq s_1$ are non-gauge and the *dual* descriptions with $0 < g_0 < s_1$ are gauge.

6.3 Physical degrees of freedom counting

Notwithstanding the simplicity of the unfolded form and the uniqueness of unfolding, there still might be a question of whether the unfolded equations do describe the correct number of physical degrees of freedom.

It is well-known that for systems with the first class constraints only, with massless fields belonging to this class, the counting of degrees of freedom is that one first level gauge parameter kills two degrees of freedom, one second level gauge parameters kills three degrees of freedom, and so on. For example, a spin-two massless field possesses $\frac{d(d-3)}{2}$ degrees of freedom, which is just $\frac{d(d+1)}{2} - 2d$, $\frac{d(d+1)}{2}$ and d being the number of components of $\phi_{\mu\nu}$ and ξ_μ.

The complete information concerning the 'number' of fields/gauge parameters, i.e., the multiplicity and the symmetry of corresponding tensors, is contained in $\mathsf{H}^k_{g=0}(\sigma_-)$ for $k = 0 \ldots p$. Not only can a number of physical degrees of freedom be calculated but the whole exact sequence that defines an $\mathfrak{iso}(d-1,1)$ *irrep* can be derived. The elements of this sequence are certain $\mathfrak{so}(d-1)$ tensors that define an $\mathfrak{so}(d-2)$ tensor as a quotient of $\mathfrak{so}(d-1)$ tensors, e.g., (4.18). The decomposition of $\mathfrak{so}(d-1,1)$-fields into irreducible tensors of $\mathfrak{so}(d-1)$ can be done with the aid of ∂_μ or, after Fourier transform, with the aid of momentum p_μ.

For example, for a spin-two field the cohomology groups that correspond to the *dynamical* fields/differential gauge parameters and its decomposition into $\mathfrak{so}(d-1)$ *irreps* are given in table 2.

Quick sort of Young diagrams in $0 \longrightarrow 2\mathsf{H}^0 \longrightarrow \mathsf{H}^1 \longrightarrow \mathcal{H}\,(0;\yng(2)) \longrightarrow 0$ gives the correct exact sequence $0 \longrightarrow \yng(1) \longrightarrow \yng(2) \longrightarrow \mathcal{H}\,(0;\yng(2)) \longrightarrow 0$, which defines a massless spin-two *irrep* $\mathcal{H}\,(0;\yng(2))$.

For the simplest mixed-symmetry field, the hook-$\yng(2,1)$, there are two levels of gauge transformation, hence, relevant cohomology groups are H^0, H^1 and H^2 (see table 3). Again,

	$\mathfrak{so}(d-1,1)$-representatives	reduction from $\mathfrak{so}(d-1,1)$ to $\mathfrak{so}(d-1)$
H^0	$\square \sim \xi_\mu$	$\square \oplus \bullet$
H^1	$\square\!\square \oplus \bullet \sim \phi_{\mu\nu},\ \phi^{\nu}{}_{\nu} \neq 0$	$\square\!\square \oplus \square \oplus 2\bullet$

Table 2: Relevant representatives of cohomology groups for massless spin-two field and its reduction to $\mathfrak{so}(d-1)$.

	$\mathfrak{so}(d-1,1)$-representatives	reduction from $\mathfrak{so}(d-1,1)$ to $\mathfrak{so}(d-1)$
H^0	$\square \sim \xi_\mu$	$\square \oplus \bullet$
H^1	$\begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square\!\square \oplus \bullet \sim \xi^A_{\mu\nu} \oplus \xi^S_{\mu\nu},\ \xi^{S\nu}_{\nu} \neq 0$	$\begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square\!\square \oplus 2\square \oplus 2\bullet$
H^2	$\begin{smallmatrix}\square\!\square\\\square\end{smallmatrix} \oplus \square \sim \phi_{\mu\nu,\lambda},\ \phi_{\mu\nu,}{}^{\nu} \neq 0$	$\begin{smallmatrix}\square\!\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square\!\square \oplus 2\square \oplus \bullet$

Table 3: Relevant representatives of cohomology groups for massless spin-$\begin{smallmatrix}\square\!\square\\\square\end{smallmatrix}$ field and its reduction to $\mathfrak{so}(d-1)$.

quick sort of diagrams in $0 \longrightarrow 3\mathsf{H}^0 \longrightarrow 2\mathsf{H}^1 \longrightarrow \mathsf{H}^2 \longrightarrow \mathcal{H}\left(0;\begin{smallmatrix}\square\!\square\\\square\end{smallmatrix}\right) \longrightarrow 0$ gives exact sequence (4.18).

Consequently, the problem of calculation the number of physical degrees of freedom, to be precise, of deriving the exact sequence, is effectively reduced to the simple combinatoric problem of (i) decomposition $\mathfrak{so}(d-1,1)$-Young diagram representatives of $\mathsf{H}(\sigma_-)$ to $\mathfrak{so}(d-1)$-diagrams; (ii) cancellation of like terms in sequence

$$0 \longrightarrow (p+1)\mathsf{H}^0 \longrightarrow p\mathsf{H}^1 \longrightarrow \ldots \longrightarrow 2\mathsf{H}^{p-1} \longrightarrow \mathsf{H}^p \longrightarrow \mathcal{H}(0;\mathbf{Y}) \longrightarrow 0. \tag{6.22}$$

Skipping combinatoric technicalities we state that in the general case of a spin-$\mathbf{Y}$ massless mixed-symmetry field (6.22) reduces to the correct exact sequence that defines a *uirrep* $\mathcal{H}(0;\mathbf{Y})$ of $\mathfrak{iso}(d-1,1)$.

6.4 Solving the generalized Jacobi identities

Unfolding some dynamical system there arises a problem of solving Jacobi identities (5.3) that have schematically a form $h \ldots h d\omega^{1\cdots}_{\mathbf{q}} \equiv 0$, where a number of background vielbeins h^a is contracted with the fiber indices of $\omega^{1\cdots}_{\mathbf{q}}$. The Jacobi identity restricts $d\omega^{1\cdots}_{\mathbf{q}}$ to have a certain specific form $d\omega^{1\cdots}_{\mathbf{q}} = h \ldots h\omega^{2\cdots}_{\mathbf{r}}$, where the $\mathfrak{so}(d-1,1)$-*irrep*, in which $\omega^{2\cdots}_{\mathbf{r}}$ takes values, the degree r and the projector built of $h \ldots h$ are completely determined. The solutions are given by[20]

[20] As was pointed out in section 5.3 nonzero traces either violate Bianchi identities or introduce mass-like terms and, therefore, are ignored.

Lemma 2. *A*. *Let* $\omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k)}$ *be a degree-q form taking values in* $\mathbf{Y} = \mathbf{Y}\{(s,p),(k,1)\}$ $\mathfrak{so}(d-1,1)$-irrep*. The general solution of*

$$h_c d\omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k-1)c} = 0 \tag{6.23}$$

has the form

$$d\omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k)} = \begin{cases} h_c\omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k)c}, & k<s, \\ h_c\dots h_c\omega_{\mathbf{q-p-1}}^{a_1(s)c,\dots,a_p(s)c,a_{p+1}(s)c}, & k=s, \quad q\geq p+1, \\ 0, & k=s, \quad q<p+1, \end{cases} \tag{6.24}$$

where $\omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k+1)}$ *and* $\omega_{\mathbf{q-p-1}}^{a_1(s+1),\dots,a_p(s+1),a_{p+1}(s+1)}$ *take values in* $\mathbf{Y}\{(s,p),(k+1,1)\}$ *and* $\mathbf{Y}\{(s+1,p+1)\}$ $\mathfrak{so}(d-1,1)$-irreps*, respectively.*[21]

Proof. The parametrization of $d\omega_{\mathbf{q}}$ by a degree-$(q+1)$ form taking values in the same $\mathfrak{so}(d-1,1)$-*irrep*, $d\omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k)} = R_{\mathbf{q+1}}^{a_1(s),\dots,a_p(s),b(k)}$, obviously fails to satisfy (6.23). Inasmuch as (6.23) contains a vielbein, the solution has to contain a number of vielbeins too. The most general parametrization of $d\omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k)}$ with only one vielbein included has the form $d\omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k)} = h_d\omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k)|d}$ for some q-form taking values in a tensor product of $\mathbf{Y}$ by a vector representation, i.e., there is no definite symmetry between index d and the rest of the indices.

$$h_ch_d\omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k-1)c|d} = 0 \longleftrightarrow \omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k-1)[c|d]} = 0. \tag{6.25}$$

Only those *irreps* in $\mathbf{Y}\otimes\square$ are allowed that have c and d symmetric, i.e., correspond to $\mathbf{Y}\{(s,p),(k+1,1)\}$, for $k<s$. This is not possible in the case $k=s$, i.e., $\mathbf{Y}=\mathbf{Y}\{(s,p+1)\}$, and the only possibility to have c and d symmetric is to add the whole column to $\mathbf{Y}$, which requires $d\omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k)}$ to be represented as $d\omega_{\mathbf{q}}^{a_1(s),\dots,a_p(s),b(k)} = h_c\dots h_c\omega_{\mathbf{q-p-1}}^{a_1(s-1)c,\dots,a_p(s-1)c,a_{p+1}(s-1)c}$ and, therefore, q must be large enough. Roughly speaking, the proof is based on the fact that the anti-symmetrization of two indices at the same row of a Young diagram is identically zero, the anti-symmetrization being due to contraction with vielbeins.

Note that choosing nonmaximal solutions of Jacobi identities results in lowering the gauge symmetry so that not all *redundant* components are excluded.

Unfolding totally-symmetric massless higher-spin fields the possible r.h.s. terms in $d\omega_{\mathbf{1}}^{a(s-1),b(t)} = \cdots$ are restricted by Jacobi identities and an essential use is made of

Corollary 1. *The solution of Jacobi identity*

$$h_cd\omega_{\mathbf{1}}^{a(s-1),b(t-1)c} \equiv 0 \tag{6.26}$$

is of the form

$$d\omega_{\mathbf{1}}^{a(s-1),b(t)} = \begin{cases} h_c\omega_{\mathbf{1}}^{a(s-1),b(t)c} & t<s-1, \\ h_ch_dC_{\mathbf{0}}^{a(s-1)c,b(s-1)d} & t=s-1. \end{cases} \tag{6.27}$$

[21] The symmetric basis is used, being more convenient in this case as the contracted with vielbeins tensors are already irreducible.

The following statement is a generalization of the Lemma-A to the case where $\{(s,p),(k,1)\}$ is a 'subdiagram' of a larger Young diagram $\mathbf{Y}$ that has a number of rows precedent/succedent to $\{(s,p),(k,1)\}$. Appropriate Young symmetrizers have to be included as the mere contraction of a number of vielbeins breaks the irreducibility of the tensor. For instance, $h_c\omega^{a(s_1),b(s_2-1)c}$ is already irreducible with the symmetry of $\mathbf{Y}\{s_1,s_2-1\}$, but this is not the case for $h_c\omega^{a(s_1-1)c,b(s_2)}$, which has to be added one term $h_c\omega^{a(s_1-1)c,b(s_2)}+\frac{1}{s_1-s_2+1}h_c\omega^{a(s_1)b,b(s_2-1)c}$ to get the symmetry of $\mathbf{Y}\{s_1-1,s_2\}$.

Lemma 3. ***B.*** *Let* $\omega_{\mathbf{q}}^{\ldots,a_1(s),\ldots,a_p(s),b(k),\ldots}$ *be a degree-q form taking values in* $\mathbf{Y}=\mathbf{Y}\{(\ldots,(s,p),(k,1),\ldots\}$ $\mathfrak{so}(d-1,1)$-irrep*, where the dots stands for the blocks in* $\mathbf{Y}$ *precedent/succedent to* $\{(s,p),(k,1)\}$*. The general solution of*

$$\Pi\left[h_c d\omega_{\mathbf{q}}^{\ldots,a_1(s),\ldots,a_p(s),b(k-1)c,\ldots}\right]=0, \tag{6.28}$$

where $\Pi[\ldots]$ *is a Young symmetrizer to* $\mathbf{Y}\{\ldots,(s,p)(k-1,1),\ldots\}$*, has the form*

$$d\omega_{\mathbf{q}}^{\ldots,a_1(s),\ldots,a_p(s),b(k),\ldots}=\begin{cases}\Pi\left[h_c\omega_{\mathbf{q}}^{\ldots,a_1(s),\ldots,a_p(s),b(k)c,\ldots}\right], & k<s,\\ \Pi\left[h_c\ldots h_c\omega_{\mathbf{q-p-1}}^{\ldots,a_1(s)c,\ldots,a_p(s)c,a_{p+1}(s)c,\ldots}\right], & k=s,\quad q\geq p+1,\\ 0, & k=s,\quad q<p+1,\end{cases} \tag{6.29}$$

where $\omega_{\mathbf{q}}^{\ldots,a_1(s),\ldots,a_p(s),b(k+1),\ldots}$ *and* $\omega_{\mathbf{q-p-1}}^{\ldots,a_1(s+1),\ldots,a_p(s+1),a_{p+1}(s+1),\ldots}$ *takes values in* $\mathbf{Y}\{\ldots,(s,p),(k+1,1),\ldots\}$ *and* $\mathbf{Y}\{\ldots,(s+1,p+1),\ldots\}$ $\mathfrak{so}(d-1,1)$-irreps*, respectively, and* $\Pi[\ldots]$ *is a Young symmetrizer to* $\mathbf{Y}\{\ldots,(s,p),(k,1),\ldots\}$.

Proof. The proof is similar to that of Lemma-A with the only comment that the symmetrizers do not affect the argumentation that anti-symmetrization of two indices in the same row is identically zero.

6.5 Dynamical content via σ_- cohomology

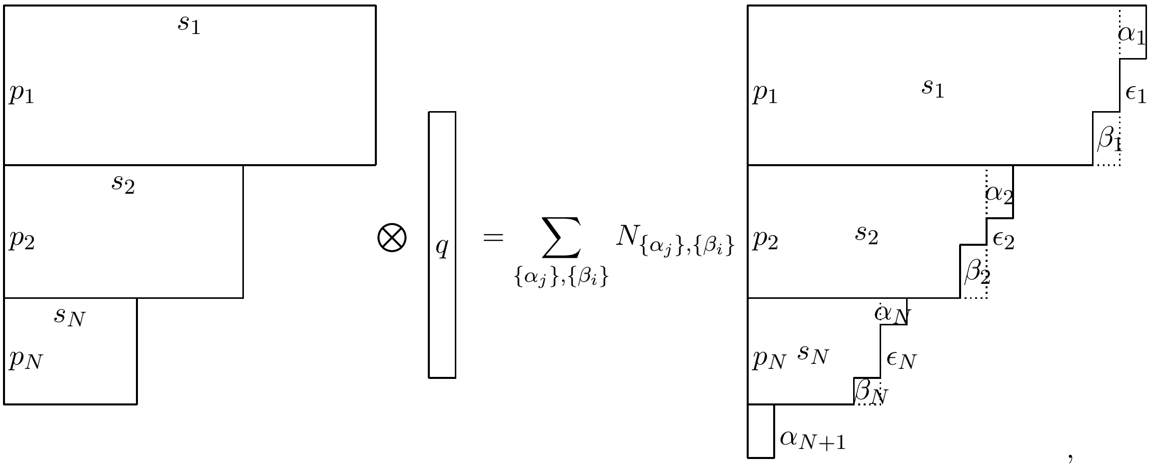
Before discussing the calculation of σ_- cohomology, let us first recall necessary facts concerning the evaluation of $\mathfrak{so}(d-1,1)$-tensor products. Let $\mathtt{P}(\epsilon_1,\ldots,\epsilon_N|m)$ be a number of integer partitions $k_1+\cdots+k_N=m$ of m, where k_i is constrained by $k_i\leq\epsilon_i$ and different rearrangements of k_i satisfying the constraints are regarded as distinct partitions. The generating function for the partition $\mathtt{P}(\epsilon_1,\ldots,\epsilon_N|m)$ is

$$\sum_m \mathtt{P}(\epsilon_1,\ldots,\epsilon_N|m)t^m=\prod_{i=1}^{i=N}\frac{(1-t^{\epsilon_i+1})}{1-t} \tag{6.30}$$

These integer partitions gives the multiplicities of *irreps* in $\mathfrak{so}(d-1,1)$-tensor products (Clebsh-Gordon coefficients) we are interested in [71].

JHEP07(2008)004

The $\mathfrak{so}(d-1,1)$-tensor product of an arbitrary *irrep* $\mathbf{Y}^* = \mathbf{Y}\{(s_i,p_i)\}$ by a one-column diagram of the height q^* can be explicitly calculated as



, (6.31)

where α_i, β_i : $\alpha_i+\beta_i \leq p_i$, $i\in[1,N]$, $\alpha_{N+1}\geq 0$ and there exist $\rho \geq 0$ such that

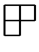
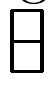
$$q^* = \sum_{i=1}^{i=N}(\alpha_i+\beta_i)+\alpha_{N+1}+2\rho. \tag{6.32}$$

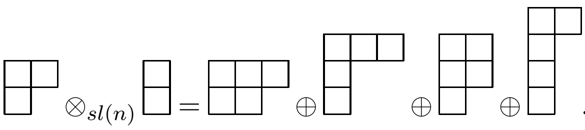
The multiplicity $N_{\{\alpha_j\},\{\beta_i\}}$ of $\mathbf{Y}^*_{\{\alpha_j,\beta_i\}}$ is given by integer partition

$$N_{\{\alpha_j\},\{\beta_i\}} = \mathtt{P}(\epsilon_1,\dots,\epsilon_N|\rho), \quad \epsilon_i = p_i-\alpha_i-\beta_i \tag{6.33}$$

and the total trace order r is

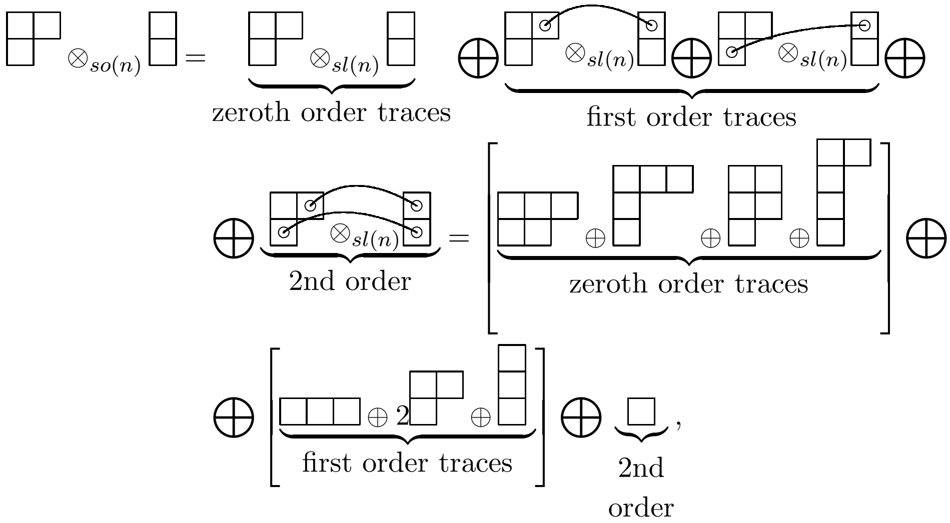
$$r = \sum_{i=1}^{i=N}\beta_i+\rho \tag{6.34}$$

Roughly speaking, to obtain an element of the tensor product one should, first, cut off from bottom-right of the i-th block a column of height $\beta_i-\gamma_i$, $p_i\geq\beta_i-\gamma_i\geq 0$ (to take different traces) and, second, add an arbitrary number $\alpha_i+\gamma_i$, $p_i\geq\alpha_i+\gamma_i\geq 0$ of cells to each block, provided the γ_i cells annihilate, i.e., they are added to the places from which γ_i cells were removed at the first stage. Multiplicity may be different from one due to different rearrangements of γ_i, i.e., when multiplied by the rest of q-column different traces can give rise to the same diagram. The number of such rearrangements is given by $\mathtt{P}(\epsilon_1,\dots,\epsilon_N|\rho)$, $\rho=\sum_{i=1}^{i=N}\gamma_i$. For instance, $sl(n)$-tensor product  $\otimes$  is given simply by



. (6.35)

The $so(d)$-tensor product can be represented as a sum of the form



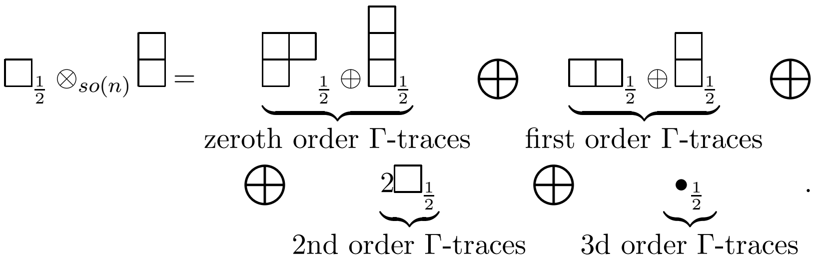
(6.36)

where the sum is over trace order and the boxes that are connected by the arc are to be contracted and, then, the $sl(n)$-product rules are to be applied to the rest of the diagrams.

Evaluating the $\mathfrak{so}(d-1,1)$-tensor product of an arbitrary spinor-*irrep* $\mathbf{Y}^* = \mathbf{Y}\{(s_i,p_i)\}_{\frac{1}{2}}$ by a one-column diagram of the height q^* one can contract any number of Γ matrices with the indices of the column and then multiply, therefore, the tensor product rule for spin-tensors can be reduced to bosonic rule (6.31) as

$$\mathbf{Y}\{(s_i,p_i)\}_{\frac{1}{2}} \otimes_{so(n)} \mathbf{Y}\{(1,q)\} = \sum_{k=0}^{q} \big(\mathbf{Y}\{(s_i,p_i)\} \otimes_{so(n)} \mathbf{Y}\{(1,q-k)\}\big)_{\frac{1}{2}}\,, \tag{6.37}$$

for example,



(6.38)

The multiplicity $N^{\frac{1}{2}}_{\{\alpha_j\},\{\beta_i\}}$ of $\mathbf{Y}^*_{\{\alpha_j,\beta_i\}\frac{1}{2}}$ is given by

$$N^{\frac{1}{2}}_{\{\alpha_j\},\{\beta_i\}} = \sum_{i=0}^{i=\rho} \mathtt{P}(\epsilon_1,\ldots,\epsilon_N|\rho-i). \tag{6.39}$$

Let us now analyze the origin of σ_- cohomology. Its action $\sigma_- : \mathcal{W}^g_{\mathbf{q}} \to \mathcal{W}^{g-1}_{\mathbf{q+1}}$ preserves $g+q$ and hence the whole complex $\mathcal{C}(\mathcal{W},\sigma_-)$ is a direct sum of complexes $\mathcal{C}(q',\sigma_-)$, where q' refers to the end element $\mathcal{W}^{g=0}_{\mathbf{q'}}$ of the complex

$$\mathcal{C}(q',\sigma_-): \qquad 0 \longrightarrow \ldots \longrightarrow \mathcal{W}^{g=1}_{\mathbf{q'-1}} \longrightarrow \mathcal{W}^{g=0}_{\mathbf{q'}} \longrightarrow 0. \tag{6.40}$$

Moreover, σ_- preserves the total rank (the form degree + rank of the $\mathfrak{so}(d-1,1)$-*irrep* in which the field takes values). Let $W^{a(s_1),\ldots,a(s_m)}_{\mathbf{q}}$ be an element of $\mathcal{W}^g_{\mathbf{q'}}$. When all form indices of $W^{a(s_1),\ldots,a(s_m)}_{\mathbf{q}}$ are converted to the fiber ones according to (5.29)

$$W^{a_1(s_1),\ldots,a_m(s_m)|[d_1\ldots d_q]} = W^{a_1(s_1),\ldots,a_m(s_m)}_{\mu_1\ldots\mu_q} h^{\mu_1 d_1}\ldots h^{\mu_q d_q}, \tag{6.41}$$

the action of σ_- is just an anti-symmetrization of all $d_1,\ldots,d_q$ with those indices a_i that are extra as compared to $\mathbf{Y}_{g-1}$ plus some terms to restore the correct Young symmetry of $\mathbf{Y}_{g-1}$. Let the decomposition of $W^{a_1(s_1),\ldots,a_m(s_m)|[d_1\ldots d_q]}$ into $\mathfrak{so}(d-1,1)$-*irreps* be of the form

$$\mathbf{Y}\{a_1(s_1),\ldots,a_m(s_m)\}\bigotimes\mathbf{Y}\{(1,q)\} = \bigoplus_{r=0}\bigoplus_{i_r} N^r_{i_r}\mathbf{Y}^r_{i_r}, \tag{6.42}$$

where the summation is over the trace order r and, then, over i_r, which enumerates all $\mathfrak{so}(d-1,1)$-*irreps* that enters in the tensor product as the traces of the r-th order, $N^r_{i_r}$ being the multiplicity of $\mathbf{Y}^r_{i_r}$. The multiplicity of the zeroth order traces is always equal to one, $N^0_{i_0}=1$. In fact, the diagrams $Y^0_{i_0}$ can be directly obtained by the $sl(n)$-tensor product rule.

The very anti-symmetrization is insensitive to whether a certain component enters as a trace or not. When decomposed into $\mathfrak{so}(d-1,1)$-*irreps* the elements of $\mathcal{W}^g_{\mathbf{q'}}$ and $\mathcal{W}^{g-1}_{\mathbf{q'+1}}$ have a number of components of the same symmetry type. The action of σ_- is just a linear transformation that either sends the whole $\mathfrak{so}(d-1,1)$-*irrep* to zero[22] or sends it to the components of $\mathcal{W}^{g-1}_{\mathbf{q'+1}}$ of the same symmetry type. Important is that σ_- does not act between different $\mathfrak{so}(d-1,1)$-*irreps*. Therefore, complex $\mathcal{C}(q',\sigma_-)$ is a direct sum of complexes, parameterized by $\mathfrak{so}(d-1,1)$-*irreps* $\mathbf{Y}'$ that are given by various tensor products

$$\mathbf{Y}_{\{n,k\}}\bigotimes\mathbf{Y}\{(1,q)\} \tag{6.43}$$

provided that the field $W^{\mathbf{Y}_{\{n,k\}}}_{\mathbf{q}}$ is an element of $\mathcal{W}^{g'-i}_{\mathbf{q'+i}}$ for certain i. When reduced to such a complex

$$\mathcal{C}(\mathbf{Y}',q',\sigma_-): \qquad 0\rightarrow\ldots\rightarrow V_g\rightarrow V_{g-1}\rightarrow\ldots\rightarrow 0 \tag{6.44}$$

the action of σ_- is a linear transformation between the spaces $V_g\rightarrow V_{g-1}$, dimensions of which are equal to the multiplicity of $\mathbf{Y}'$ in the decomposition of $W^{Y_g}_{\mathbf{q_g}}$ and $W^{Y_{g-1}}_{\mathbf{q_g+1}}$. The action of σ_- is maximally non-degenerate compatible with nilpotency. Therefore, to find cohomology groups the dimension of each linear space in every complex has to be calculated.

Let us note, that only the types and multiplicities of $\mathfrak{so}(d-1,1)$-*irreps* are found below, i.e., no attention is paid to the description of how the corresponding tensors are contained in the fields $W^{\mathbf{Y}}_{\mathbf{q'}}$. This is sufficient for our purposes, though, it is obvious (6.12) how the *metric-like* field $\phi_{\mathbf{Y}_M}$ is incorporated in an element of $\mathcal{W}^{g=0}_{\mathbf{p}}$.

The calculation of σ_- cohomology is divided into three cases that cover the whole variety of $\mathfrak{so}(d-1,1)$-*irreps* that can result from the tensor products (6.43).

[22]provided that the appropriate basis on the space of $\mathfrak{so}(d-1,1)$-*irreps* with the same symmetry type is chosen.

In the Lemmas below it is assumed that complex $\mathcal{C}(\mathbf{Y}', q', \sigma_-)$ is parameterized by $\mathfrak{so}(d-1,1)$-*irrep* $\mathbf{Y}' = \mathbf{Y}^*_{\{\alpha_j,\beta_i\}}$ of the form (6.31) with $\mathbf{Y}^* = \mathbf{Y}_{\{n,k\}}$, $q^* = q$ such that $W_{\mathbf{q}}^{\mathbf{Y}_{\{n,k\}}} \in \mathcal{C}(q', \sigma_-)$ and ρ is defined according to (6.32).

In the first case $\mathbf{Y}'$ is an element of $\mathbf{Y}_0 \otimes \mathbf{Y}\{(1,q)\}$

Lemma 4. *The σ_- cohomology groups $\mathsf{H}^q_{g=0}$ and $\mathsf{H}^q_{g=1}$ are nontrivial and are given by*

$$\mathsf{H}^q_g = \begin{cases} \mathbf{Y}\{(s_i-1,p_i)\}_{\{\alpha_j,\beta_i\}} M_{\{\alpha_j,\beta_i\}} : \begin{array}{c}\alpha_{N+1}=0,\\ \sum_{i=0}^{i=N}\alpha_i = q,\end{array} & q\in[0,p], \quad g=0,\\ \mathsf{H}^{2p-q-1}, & q\in[p+1,2p+1], \quad g=1,\\ \emptyset, & q>2p+1, \quad \textit{any } g, \end{cases} \tag{6.45}$$

where $M_{\{\alpha_j,\beta_i\}}$ is the multiplicity of the irrep $\mathbf{Y}\{(s_i-1,p_i)\}_{\{\alpha_j,\beta_i\}}$, *defined below.*

Proof. From the very form of $\mathbf{Y}_{\{n=N,0\}} \equiv \mathbf{Y}_{g=0} \equiv \mathbf{Y}\{(s_i-1,p_i)\}$ it follows that the $\mathfrak{so}(d-1,1)$-*irreps* that the element of $\mathcal{W}^{g=0}_{\mathbf{q}} \sim \mathbf{Y}_0 \otimes \mathbf{Y}\{(1,q)\}$ decomposes into may be contained in $\mathcal{W}^{g=1}_{\mathbf{q-1}}$ only, i.e., $\mathcal{W}^{g=0}_{\mathbf{q}}$ and $\mathcal{W}^{g=k}_{\mathbf{q-k}}$ contain no $\mathfrak{so}(d-1,1)$-*irreps* of the same symmetry type for $k>1$. Therefore, the length of complex $\mathcal{C}(\mathbf{Y}', q, \sigma_-)$, where $\mathbf{Y}' \in \mathbf{Y}_0 \otimes \mathbf{Y}\{(1,q)\}$ is equal to one, i.e., $\mathcal{C}(\mathbf{Y}\{(s_i-1,p_i)\}_{\{\alpha_j,\beta_i\}}, q, \sigma_-) : 0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow 0$, where the dimensions of V_0 and V_1 are given by the multiplicities of the $\mathfrak{so}(d-1,1)$-*irrep* $\mathbf{Y}\{(s_i-1,p_i)\}_{\{\alpha_j,\beta_i\}}$ in $\mathcal{W}^{g=1}_{\mathbf{q-1}}$ and $\mathcal{W}^{g=0}_{\mathbf{q}}$, respectively,

$$\begin{aligned} \dim(V_1) &= \begin{cases} \begin{cases} \mathsf{P}(\epsilon_1,\dots,\epsilon_N|\rho-1) & \rho\geq 1,\\ 0, & \rho=0, \end{cases} & \alpha_{N+1}=0,\\ \mathsf{P}(\epsilon_1,\dots,\epsilon_N|\rho), & \alpha_{N+1}>0, \end{cases}\\ \dim(V_0) &= \mathsf{P}(\epsilon_1,\dots,\epsilon_N|\rho), \end{aligned} \tag{6.46}$$

If $\alpha_{N+1}>0$ the dimensions of V_0 and V_1 are equal and each irreducible component of $\mathcal{W}^{g=0}_{\mathbf{q}}$ with $\alpha_{N+1}>0$ can be gauge away by virtue of the corresponding element of $\mathcal{W}^{g=1}_{\mathbf{q-1}}$, thus being exact. $\mathsf{H}^{k>2p+1} = \emptyset$, inasmuch as $\rho \leq p$ and the components of the forms with rank greater than $(2p+1)$ must have $\alpha_{N+1}>0$, thus being exact. If $\alpha_{N+1}=0$ the dimensions are different, $\dim(V_1) < \dim(V_0)$ for $q\leq p$, $\dim(V_1) = \dim(V_0)$ for $q=p+1$ and $\dim(V_1) > \dim(V_0)$ for $q>p$. Therefore, the number of those $\mathbf{Y}\{s_i,p_i\}_{\{\alpha_j,\beta_i\}} \in \mathcal{W}^{g=0}_{\mathbf{q}}$, $q\leq p$ that are not exact is equal to $M_{\{\alpha_j,\beta_i\}} = |\dim(V_1) - \dim(V_0)|$, the same is the number of those $\mathbf{Y}\{s_i,p_i\}_{\{\alpha_j,\beta_i\}} \in \mathcal{W}^{g=1}_{\mathbf{q-1}}$, $q>p+1$ that are not exact. The obvious property of integer partitions

$$\mathsf{P}(\epsilon_1,\dots,\epsilon_N|m) = \mathsf{P}(\epsilon_1,\dots,\epsilon_N|\epsilon_1+\dots+\epsilon_N-m) \tag{6.47}$$

results in the important duality in the cohomology groups $\mathsf{H}^{p-k}_{r,g=0} \sim \mathsf{H}^{p+k+1}_{r+k+1,g=1}$ or, roughly speaking, $\mathsf{H}^p \sim \mathsf{H}^{p+1}$, $\mathsf{H}^{p-1} \sim \mathsf{H}^{p+2}$ and so on.

The second case is the complex $\mathcal{C}(\mathbf{Y}', q', \sigma_-)$, where $\mathbf{Y}'$ is an element of the tensor product $\mathbf{Y}_{\{n,k\}} \otimes \mathbf{Y}\{(1,q)\}$ for certain $q\geq 0$ with $1<k<k_n^{max}-1$.

Lemma 5. *If $\mathbf{Y}'$ is an element of the tensor product $\mathbf{Y}_{\{n,k\}} \otimes \mathbf{Y}\{(1,q)\}$ for certain $q' \geq 0$ with $1 < k < k_n^{max} - 1$, the complex $\mathcal{C}(\mathbf{Y}', q', \sigma_-)$ is acyclic.*

Proof. The complex has the length two, i.e., $0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow 0$, where the dimensions are given by $\dim(V_0) = \mathtt{P}(\epsilon_1, \dots, \epsilon_N|\rho)$, $\dim(V_1) = \mathtt{P}(\epsilon_1, \dots, \epsilon_N, 1|\rho+1)$, $\dim(V_2) = \mathtt{P}(\epsilon_1, \dots, \epsilon_N|\rho+1)$. Simple calculations with generating functions results in $\dim(V_0) < \dim(V_1)$, $\dim(V_2) < \dim(V_1)$ and $\dim(V_0) - \dim(V_1) + \dim(V_2) = 0$ and, consequently, the sequence is exact.

Analogously,

Lemma 6. *If $\mathbf{Y}'$ is an element of the tensor product $\mathbf{Y}_{\{n,k\}} \otimes \mathbf{Y}\{(1,q)\}$ for certain $q \geq 0$ with $k = 0$ and $n < N$, the complex $\mathcal{C}(\mathbf{Y}', q', \sigma_-)$ is acyclic.*

Proof. The speciality of such $\mathbf{Y}'$ is that the complex has the length three, i.e., $0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$, where the dimensions are given by

$$\begin{aligned}
&\dim(V_0) = \mathtt{P}(\epsilon_1, \dots, \epsilon_N|\rho), \\
&\dim(V_1) = \mathtt{P}(\epsilon_1, \dots, \epsilon_{n-1}, \epsilon_n + 1, \epsilon_{n+1}, \dots, \epsilon_N|\rho), \\
&\dim(V_2) = \begin{cases} \mathtt{P}(\epsilon_1, \dots, \epsilon_{n-1}, p_n - \epsilon_n, \epsilon_{n+1}, \dots, \epsilon_N|\rho - \epsilon_n - 1) & \rho \geq \epsilon_n + 1, \\ 0, & \text{otherwise}, \end{cases}, \\
&\dim(V_3) = \begin{cases} \mathtt{P}(\epsilon_1, \dots, \epsilon_{n-1}, p_n - \epsilon_n - 1, \epsilon_{n+1}, \dots, \epsilon_N|\rho - \epsilon_n - 2) & \rho \geq \epsilon_n + 2, \\ 0, & \text{otherwise}, \end{cases},
\end{aligned}$$

Again, simple calculations with generating functions results in appropriate inequalities and $\dim(V_0) - \dim(V_1) + \dim(V_2) - \dim(V_3) = 0$ and, consequently, the sequence is exact.

The above three cases covers the whole variety of the $\mathfrak{so}(d-1,1)$-tensors that can result from the tensor products $\mathbf{Y}_{\{n,k\}} \otimes \mathbf{Y}\{(1,q)\}$ for any n, k and q. The cases with $s_N = 1$ and $\mathbf{Y} = \mathbf{Y}\{0\}$ are special but the calculation of cohomology groups leads to the same result.

In the fermionic case, the computations are similar due to the multiplicity given by (6.39). The difference is that all nontrivial cohomology is concentrated in grade zero. Indeed, taking into account (6.46), where $\mathtt{P}(\epsilon_1, \dots, \epsilon_N|\rho)$ is to be replaced by (6.39), it follows that $\dim(V_0) \geq \dim(V_1)$ for all q and $\mathsf{H}^{p-k}_{r,g=0} \sim \mathsf{H}^{p+k+1}_{r+k+1,g=0}$, where r is referred to the Γ-trace order, or, roughly speaking, $\mathsf{H}^p \sim \mathsf{H}^{p+1}$, $\mathsf{H}^{p-1} \sim \mathsf{H}^{p+2}$ and so on.

7. Conclusions

The unfolded system constructed in the paper has a simple form of a covariant constancy equation and describes arbitrary mixed-symmetry bosonic and fermionic massless fields in d-dimensional Minkowski space. The gauge fields/parameters are differential forms with values in certain finite-dimensional irreducible representations of the Lorentz algebra that are uniquely determined by the generalized spin.

The key moment is that all gauge symmetries are manifest within the unfolded formulation, which is of most importance in controlling the number of physical degrees of

JHEP07(2008)004

freedom when trying to introduce interactions. Unfolded systems are formulated in terms of differential forms, which is a natural way to respect diffeomorphisms and, hence, to describe systems that include gravity.

Though, the necessary conditions for the system to have a lagrangian description are satisfied, i.e., the fields are in one-to-one correspondence with the equations and the k-th level gauge symmetries are in one-to-one correspondence with the k-th Bianchi identities, the very lagrangian remains to be constructed.

Another interesting moment is that the unfolded equations for bosons are the same as for fermions, namely, the operators involved, i.e., exterior differential d and σ_-, remains unmodified when tensors are replaced with spin-tensors. Though this nice property partly breaks down in (anti)-de Sitter space, within the unfolded approach bosons and fermions have much in common, the fact being very useful for supersymmetric theories.

The proposed unfolded system includes all non-gauge *dual* descriptions, which are based on the generalized Weyl tensor and its descendants, but not all of gauge *dual* descriptions. It would be interesting to construct an unfolded system that contains all *dual* formulations.

The interactions of the totally symmetric massless higher-spin fields are known to require a nonzero cosmological constant, i.e., are formulated in (anti)-de Sitter space [2]. Mixed-symmetry fields exhibit some interesting features in the presence of cosmological constant. For example, not all of the Minkowski gauge symmetries can be deformed to (anti)-de Sitter [72]. As a result massless mixed-symmetry fields have more degrees of freedom in (anti)-de Sitter compared to its Minkowski counterparts [73] and in the Minkowski limit a massless mixed-symmetry field splits in a certain collection of massless fields, generally. Contrariwise, a single mixed-symmetry field can not be smoothly deformed to (anti)-de Sitter. Another interesting effect is the existence of the so-called partially-massless fields [74 − 84], the fields that have a number of degrees of freedom intermediate between that of massless and massive and split in a set of massless fields in the Minkowski limit. Therefore, the extension of the proposed approach to (anti)-de Sitter space seems to be non-trivial but nevertheless worth being investigated.

In a series of papers [85, 61, 86, 60] it was suggested so-called *frame-like* approach to the general mixed-symmetry fields in (anti)-de Sitter. To generalize the proposed in the present paper unfolded system to (anti)-de Sitter case and to compare to that of [60] is the next step to perform.

We consider the proposed unfolded system as the first stage in constructing the full interacting theory of mixed-symmetry fields.

Acknowledgments

The author appreciates sincerely M.A. Vasiliev for reading the manuscript and making many detailed and valuable suggestions and illuminating comments, the very work was initiated as the result of discussions with M.A. Vasiliev. The author is also grateful to R.R. Metsaev and K.B. Alkalaev for useful discussions and to O.V. Shaynkman for discussion of the fermionic case. The work was supported in part by grants RFBR No. 05-02-17654, LSS No. 4401.2006.2 and INTAS No. 05-7928, by the Landau Scholarship and by the Scholarship of Dynasty foundation.

A. Multi-index convention

The multi-index notations is used: a group of indices in which certain tensor is symmetric or is to be symmetrized is denoted either by one letter with the number of indices indicated in round brackets, or by placing a group of indices in round brackets, e.g.,

$$T^{a(s)} \equiv T^{a_1 a_2 \ldots a_s}: \qquad T^{a_1 \ldots a_i \ldots a_j \ldots a_s} = T^{a_1 \ldots a_j \ldots a_i \ldots a_s} \tag{A.1}$$

$$V^a T^{a(s)} \equiv V^{(a_1} T^{a_2 \ldots a_{s+1})} \equiv \frac{1}{s+1} \left(V^{a_1} T^{a_2 a_3 \ldots a_{s+1}} + V^{a_2} T^{a_1 a_3 \ldots a_{s+1}} + \cdots + V^{a_{s+1}} T^{a_1 a_2 \ldots a_s}\right) \tag{A.2}$$

$$V^{(b} T^{a(s))} \equiv V^{(b} T^{a_1 \ldots a_s)} \equiv \frac{1}{s+1} \left(V^b T^{a_1 a_2 \ldots a_s} + V^{a_1} T^{b a_2 a_3 \ldots a_s} + \cdots + V^{a_s} T^{b a_1 a_2 \ldots a_{s-1}}\right) \tag{A.3}$$

Analogously, the group of indices in which certain tensor is anti-symmetric or is to be anti-symmetrized is denoted by placing indices in square brackets, e.g.,

$$T^{a[s]} \equiv T^{a_1 a_2 \ldots a_s}: \qquad T^{a_1 \ldots a_i a_{i+1} \ldots a_s} = -T^{a_1 \ldots a_{i+1} a_i \ldots a_s} \tag{A.4}$$

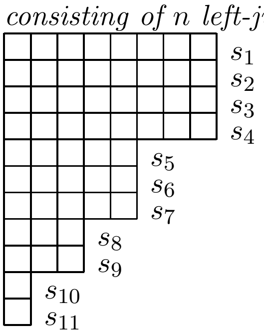
$$V^{[b} T^{a[s]]} \equiv V^{[b} T^{a_1 \ldots a_s]} \equiv \frac{1}{s+1} \left(V^b T^{a_1 a_2 \ldots a_s} - V^{a_1} T^{b a_2 a_3 \ldots a_s} + \cdots + (-)^s V^{a_s} T^{b a_1 a_2 \ldots a_{s-1}}\right) \tag{A.5}$$

The operators of (anti)-symmetrization are weighted to be projectors (the factor $\frac{1}{s+1}$ above).

B. Young diagrams

Comprehensive information on Young diagrams can be found, for example, in the textbook [71].

Definition 1. *Given an integer partition, i.e., a nonincreasing sequence $\{s_i, i \in [1,n]\}$, $s_i \geq s_{i+1}$ of positive integers (or nonnegative when it is convenient to work with a sequence of a fixed length), associated Young diagram $\mathbf{Y}\{s_1, s_2, \ldots, s_n\}$ is a graphical representation consisting of n left-justified rows made of boxes, with the i-th row containing s_i boxes.*



JHEP07(2008)004

Finite-dimensional irreducible representation(*irrep*) of $sl(d)$, i.e., various irreducible $sl(d)$-tensors, are in one-to-one correspondence with Young diagrams of the form $\mathbf{Y}\{s_1, s_2, \dots, s_{[\frac{d}{2}]}\}$. The associated irreducible tensors

$$T^{\overbrace{a\dots a}^{s_1},\overbrace{b\dots b}^{s_2},\dots} \tag{B.1}$$

or, in condensed notation, $T^{a(s_1),b(s_2),\dots}$ have at most $[\frac{d}{2}]$ groups of indices, being symmetric in each group separately, and satisfy the condition that the symmetrization of any group of indices with one index of any of the subsequent groups is identically zero, i.e.,

$$T^{a(s_1),\dots,(b(s_k),\dots,b)c(s_j-1),\dots} \equiv 0, \qquad k < j. \tag{B.2}$$

If $s_{[\frac{d}{2}]} \neq 0$ and d is even the (anti-)selfduality condition has to be imposed for appropriate signature. (anti)-selfduality is conventionally denoted by the sign factors $+(-)$ before $s_{[\frac{d}{2}]}$. In this paper we do not consider (anti)-self dual representations.

A scalar representation $\mathbf{Y}\{0,0,\dots,0\}$ is denoted by $\bullet$, a vector *irrep* $\mathbf{Y}\{1,0,\dots,0\}$ by $\square$, rank-two symmetric tensor *irrep* by $\square\!\square$, rank-two antisymmetric tensor *irrep* by $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and so on.

Finite-dimensional irreducible representations of $so(d)$ are of the two types: tensor and spin-tensor; and are also characterized by Young diagrams, which in the case of spin-tensor *irreps* refer to the symmetry of the tensor part. Young diagrams that correspond to spin-tensor *irreps* are labeled by $\frac{1}{2}$-subscript. Spinor indices $\alpha, \beta, \gamma = 1 \dots 2^{[\frac{d}{2}]}$ are placed first and are separated from tensor indices by ";". For example, a spinor ψ^α *irrep* is denoted by $\bullet_{\frac{1}{2}}$, a vector-spinor *irrep* $A^{\alpha;a}$by $\square_{\frac{1}{2}}$ and so on. To make tensors irreducible, in addition to the Young symmetry condition, the tracelessness condition with respect to each pair of indices is to be imposed

$$\eta_{cc} T^{a(s_1),\dots,cb(s_i-1),\dots,cd(s_j-1),\dots,f(s_n)} \equiv 0, \qquad i = 1\dots n, \quad j = 1\dots n, \tag{B.3}$$

To make spin-tensors irreducible, in addition to the Young symmetry condition, Γ-tracelessness condition with respect to each tensor index and a spinor index is to be imposed

$$\Gamma^{\alpha}_{c\,\beta} T^{\beta;a(s_1),\dots,cb(s_i-1),\dots,f(s_n)} \equiv 0, \qquad i = 1\dots n, \tag{B.4}$$

where $\Gamma^{\alpha}_{c\,\beta}$ satisfy $\Gamma^{\alpha}_{a\,\beta}\Gamma^{\beta}_{b\,\gamma} + \Gamma^{\alpha}_{b\,\beta}\Gamma^{\beta}_{a\,\gamma} = 2\eta_{ab}$. Additional conditions on spinors, viz., Majorana, Weyl and Majorana-Weyl are irrelevant to the problems concerned. Also, in both cases it is required for the sum of the heights of the first two columns of Young diagrams to be not greater than d. Note that the Γ-tracelessness condition is stronger than the tracelessness one and applying the Γ-tracelessness twice to two symmetric indices gives the tracelessness

$$0 = 2\Gamma^{\alpha}_{a\,\beta}\Gamma^{\beta}_{b\,\gamma} T^{\gamma;(ab)} = \eta_{ab} T^{\alpha;ab}. \tag{B.5}$$

To handle with arbitrary large Young diagrams the so-called block representation is used, i.e., the rows of equal lengths are combined to blocks

$$\mathbf{Y}\{(s_1,p_1),\dots,(s_n,p_n)\} \equiv \mathbf{Y}\{\overbrace{s_1,\dots,s_1}^{p_1},\overbrace{s_2,\dots,s_2}^{p_2},\dots,\overbrace{s_n,\dots,s_n}^{p_n}\}. \tag{B.6}$$

JHEP07(2008)004

References

[1] C. Fronsdal, *Massless fields with integer spin*, *Phys. Rev.* **D 18** (1978) 3624.

[2] E.S. Fradkin and M.A. Vasiliev, *Cubic interaction in extended theories of massless higher spin fields*, *Nucl. Phys.* **B 291** (1987) 141.

[3] E.S. Fradkin and M.A. Vasiliev, *On the gravitational interaction of massless higher spin fields*, *Phys. Lett.* **B 189** (1987) 89.

[4] M.A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3 + 1)-dimensions*, *Phys. Lett.* **B 243** (1990) 378.

[5] D. Francia and A. Sagnotti, *On the geometry of higher-spin gauge fields*, *Class. and Quant. Grav.* **20** (2003) S473 [hep-th/0212185].

[6] D. Francia and A. Sagnotti, *Free geometric equations for higher spins*, *Phys. Lett.* **B 543** (2002) 303 [hep-th/0207002].

[7] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in $(A)dS_d$*, *Phys. Lett.* **B 567** (2003) 139 [hep-th/0304049].

[8] X. Bekaert, I.L. Buchbinder, A. Pashnev and M. Tsulaia, *On higher spin theory: strings, BRST, dimensional reductions*, *Class. and Quant. Grav.* **21** (2004) S1457 [hep-th/0312252].

[9] D. Sorokin, *Introduction to the classical theory of higher spins*, *AIP Conf. Proc.* **767** (2005) 172 [hep-th/0405069].

[10] X. Bekaert, S. Cnockaert, C. Iazeolla and M.A. Vasiliev, *Nonlinear higher spin theories in various dimensions*, hep-th/0503128.

[11] A. Sagnotti, E. Sezgin and P. Sundell, *On higher spins with a strong* $\mathrm{Sp}(2,R)$ *condition*, hep-th/0501156.

[12] P.K. Townsend, *Covariant quantization of antisymmetric tensor gauge fields*, *Phys. Lett.* **B 88** (1979) 97.

[13] E. Sezgin and P. van Nieuwenhuizen, *Renormalizability properties of antisymmetric tensor fields coupled to gravity*, *Phys. Rev.* **D 22** (1980) 301.

[14] M. Blau and G. Thompson, *Topological gauge theories of antisymmetric tensor fields*, *Ann. Phys. (NY)* **205** (1991) 130.

[15] G. Bonelli, *On the tensionless limit of bosonic strings, infinite symmetries and higher spins*, *Nucl. Phys.* **B 669** (2003) 159 [hep-th/0305155].

[16] T. Curtright, *Generalized gauge fields*, *Phys. Lett.* **B 165** (1985) 304.

[17] C.S. Aulakh, I.G. Koh and S. Ouvry, *Higher spin fields with mixed symmetry*, *Phys. Lett.* **B 173** (1986) 284.

[18] S. Ouvry and J. Stern, *Gauge fields of any spin and symmetry*, *Phys. Lett.* **B 177** (1986) 335.

[19] J.M.F. Labastida and T.R. Morris, *Massless mixed symmetry bosonic free fields*, *Phys. Lett.* **B 180** (1986) 101.

[20] J.M.F. Labastida, *Massless bosonic free fields*, *Phys. Rev. Lett.* **58** (1987) 531.

[21] J.M.F. Labastida, *Massless particles in arbitrary representations of the Lorentz group*, *Nucl. Phys.* **B 322** (1989) 185.

[22] X. Bekaert and N. Boulanger, *Tensor gauge fields in arbitrary representations of GL(D,R): duality and Poincaré lemma*, *Commun. Math. Phys.* **245** (2004) 27 [hep-th/0208058].

[23] X. Bekaert and N. Boulanger, *Tensor gauge fields in arbitrary representations of GL(D,R). II: quadratic actions*, *Commun. Math. Phys.* **271** (2007) 723 [hep-th/0606198].

[24] C. Burdik, A. Pashnev and M. Tsulaia, *On the mixed symmetry irreducible representations of the Poincaré group in the BRST approach*, *Mod. Phys. Lett.* **A 16** (2001) 731 [hep-th/0101201].

[25] C. Burdik, A. Pashnev and M. Tsulaia, *The Lagrangian description of representations of the Poincaré group*, *Nucl. Phys.* **102** *(Proc. Suppl.)* (2001) 285 [hep-th/0103143].

[26] P.Y. Moshin and A.A. Reshetnyak, *BRST approach to Lagrangian formulation for mixed-symmetry fermionic higher-spin fields*, *JHEP* **10** (2007) 040 [arXiv:0707.0386].

[27] I.L. Buchbinder, V.A. Krykhtin and H. Takata, *Gauge invariant Lagrangian construction for massive bosonic mixed symmetry higher spin fields*, *Phys. Lett.* **B 656** (2007) 253 [arXiv:0707.2181].

[28] Y.M. Zinoviev, *On massive mixed symmetry tensor fields in Minkowski space and (A)dS*, hep-th/0211233.

[29] Y.M. Zinoviev, *First order formalism for mixed symmetry tensor fields*, hep-th/0304067.

[30] Y.M. Zinoviev, *First order formalism for massive mixed symmetry tensor fields in Minkowski and (A)dS spaces*, hep-th/0306292.

[31] M.A. Vasiliev, *Equations of motion of interacting massless fields of all spins as a free differential algebra*, *Phys. Lett.* **B 209** (1988) 491.

[32] M.A. Vasiliev, *Consistent equations for interacting massless fields of all spins in the first order in curvatures*, *Ann. Phys. (NY)* **190** (1989) 59.

[33] M.A. Vasiliev, *Unfolded representation for relativistic equations in* $(2+1)$ *anti-de Sitter space*, *Class. and Quant. Grav.* **11** (1994) 649.

[34] R.R. Metsaev, *Cubic interaction vertices of totally symmetric and mixed symmetry massless representations of the Poincaré group in* $D = 6$ *space-time*, *Phys. Lett.* **B 309** (1993) 39.

[35] R.R. Metsaev, *Cubic interaction vertices for massive and massless higher spin fields*, *Nucl. Phys.* **B 759** (2006) 147 [hep-th/0512342].

[36] R.R. Metsaev, *Cubic interaction vertices for fermionic and bosonic arbitrary spin fields*, arXiv:0712.3526.

[37] E.P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, *Ann. Math.* **40** (1939) 149 [*Nucl. Phys.* **6** *(Proc. Suppl.)* (1989) 9].

[38] X. Bekaert and N. Boulanger, *The unitary representations of the Poincaré group in any spacetime dimension*, hep-th/0611263.

[39] X. Bekaert and J. Mourad, *The continuous spin limit of higher spin field equations*, *JHEP* **01** (2006) 115 [hep-th/0509092].

[40] L.P.S. Singh and C.R. Hagen, *Lagrangian formulation for arbitrary spin.* 1. *The boson case*, *Phys. Rev.* **D 9** (1974) 898.

JHEP07(2008)004

[41] V.I. Ogievetsky and I.V. Polubarinov, *The notoph and its possible interactions*, *Sov. J. Nucl. Phys.* **4** (1967) 156 [*Yad. Fiz.* **4** (1966) 216].

[42] T.L. Curtright and P.G.O. Freund, *Massive dual fields*, *Nucl. Phys.* **B 172** (1980) 413.

[43] N. Boulanger, S. Cnockaert and M. Henneaux, *A note on spin-s duality*, *JHEP* **06** (2003) 060 [hep-th/0306023].

[44] Y.M. Zinoviev, *On dual formulation of gravity. II: metric and affine connection*, hep-th/0506217.

[45] Y.M. Zinoviev, *On dual formulation of gravity*, hep-th/0504210.

[46] Y.M. Zinoviev, *On dual formulations of massive tensor fields*, *JHEP* **10** (2005) 075 [hep-th/0504081].

[47] X. Bekaert, N. Boulanger and M. Henneaux, *Consistent deformations of dual formulations of linearized gravity: a no-go result*, *Phys. Rev.* **D 67** (2003) 044010 [hep-th/0210278].

[48] S. Deser, P.K. Townsend and W. Siegel, *Higher rank representations of lower spin*, *Nucl. Phys.* **B 184** (1981) 333.

[49] P.K. Townsend, *Classical properties of antisymmetric tensor gauge fields*, lecture given at 18$^{\text{th}}$ *Winter School of Theoretical Physics*, Karpacz Poland (1981).

[50] P.K. Townsend, *Gauge invariance for spin* 1/2, *Phys. Lett.* **B 90** (1980) 275.

[51] P.A.M. Dirac, *Relativistic wave equations*, *Proc. Roy. Soc. Lond.* **A 155** (1936) 447.

[52] R.R. Metsaev, *Arbitrary spin massless bosonic fields in d-dimensional anti-de Sitter space*, hep-th/9810231.

[53] E.D. Skvortsov and M.A. Vasiliev, *Transverse invariant higher spin fields*, hep-th/0701278.

[54] J. Fang and C. Fronsdal, *Massless fields with half integral spin*, *Phys. Rev.* **D 18** (1978) 3630.

[55] D. Sullivan, *Infinitesimal computations in topology*, *Inst. Hautes Etudes Sci. Publ. Math.* **47** (1977) 269.

[56] R. D'Auria and P. Fré, *Geometric supergravity in* $D = 11$ *and its hidden supergroup*, *Nucl. Phys.* **B 201** (1982) 101 [*Erratum ibid.* **B 206** (1982) 496].

[57] R. D'Auria, P. Fré, P.K. Townsend and P. van Nieuwenhuizen, *Invariance of actions, rheonomy and the new minimal* $N = 1$ *supergravity in the group manifold approach*, *Ann. Phys. (NY)* **155** (1984) 423.

[58] P. van Nieuwenhuizen, *Free graded differential superalgebras*, invited talk given at 11$^{\text{th}}$ *Int. Colloq. on Group Theoretical Methods in Physics*, Istanbul Turkey (1982).

[59] M.A. Vasiliev, *On conformal,* $\mathrm{SL}(4,R)$ *and* $\mathrm{Sp}(8,R)$ *symmetries of* 4*d massless fields*, *Nucl. Phys.* **B 793** (2008) 469 [arXiv:0707.1085].

[60] K.B. Alkalaev, O.V. Shaynkman and M.A. Vasiliev, *Frame-like formulation for free mixed-symmetry bosonic massless higher-spin fields in* AdS_d, hep-th/0601225.

[61] K.B. Alkalaev, O.V. Shaynkman and M.A. Vasiliev, *On the frame-like formulation of mixed-symmetry massless fields in* $(A)dS_d$, *Nucl. Phys.* **B 692** (2004) 363 [hep-th/0311164].

[62] O.V. Shaynkman and M.A. Vasiliev, *Scalar field in any dimension from the higher spin gauge theory perspective*, *Theor. Math. Phys.* **123** (2000) 683 [hep-th/0003123].

[63] O.A. Gelfond and M.A. Vasiliev, *Higher rank conformal fields in the* $\mathrm{Sp}(2M)$ *symmetric generalized space-time*, *Theor. Math. Phys.* **145** (2005) 1400 [hep-th/0304020].

[64] V.E. LoPatin and M.A. Vasiliev, *Free massless bosonic fields of arbitrary spin in d-dimensional de Sitter space*, *Mod. Phys. Lett.* **A 3** (1988) 257.

[65] M.A. Vasiliev, *Actions, charges and off-shell fields in the unfolded dynamics approach*, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 37 [hep-th/0504090].

[66] M.A. Vasiliev, *Cubic interactions of bosonic higher spin gauge fields in* AdS_5, *Nucl. Phys.* **B 616** (2001) 106 [hep-th/0106200].

[67] T. Curtright, *Massless field supermultiplets with arbitrary spin*, *Phys. Lett.* **B 85** (1979) 219.

[68] A.S. Matveev and M.A. Vasiliev, *On dual formulation for higher spin gauge fields in* $(A)dS_d$, *Phys. Lett.* **B 609** (2005) 157 [hep-th/0410249].

[69] M.A. Vasiliev, *'Gauge' form of description of massless fields with arbitrary spin. (In russian)*, *Yad. Fiz.* **32** (1980) 855.

[70] M.A. Vasiliev, *Free massless fermionic fields of arbitrary spin in d-dimensional de Sitter space*, *Nucl. Phys.* **B 301** (1988) 26.

[71] A.O. Barut and R. Raczka, *Theory of group representations and applications*, World Scientific, Singapore (1986).

[72] R.R. Metsaev, *Massless mixed symmetry bosonic free fields in d-dimensional anti-de Sitter space-time*, *Phys. Lett.* **B 354** (1995) 78.

[73] L. Brink, R.R. Metsaev and M.A. Vasiliev, *How massless are massless fields in* AdS_d, *Nucl. Phys.* **B 586** (2000) 183 [hep-th/0005136].

[74] S. Deser and R.I. Nepomechie, *Gauge invariance versus masslessness in de Sitter space*, *Ann. Phys. (NY)* **154** (1984) 396.

[75] S. Deser and R.I. Nepomechie, *Anomalous propagation of gauge fields in conformally flat spaces*, *Phys. Lett.* **B 132** (1983) 321.

[76] A. Higuchi, *Symmetric tensor spherical harmonics on the N sphere and their application to the de Sitter group* $\mathrm{SO}(N, 1)$, *J. Math. Phys.* **28** (1987) 1553.

[77] A. Higuchi, *Forbidden mass range for spin-2 field theory in de Sitter space-time*, *Nucl. Phys.* **B 282** (1987) 397.

[78] A. Higuchi, *Massive symmetric tensor field in space-times with a positive cosmological constant*, *Nucl. Phys.* **B 325** (1989) 745.

[79] S. Deser and A. Waldron, *Partial masslessness of higher spins in (A)dS*, *Nucl. Phys.* **B 607** (2001) 577 [hep-th/0103198].

[80] S. Deser and A. Waldron, *Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations*, *Phys. Lett.* **B 513** (2001) 137 [hep-th/0105181].

[81] Y.M. Zinoviev, *On massive high spin particles in (A)dS*, hep-th/0108192.

[82] S. Deser and A. Waldron, *Conformal invariance of partially massless higher spins*, *Phys. Lett.* **B 603** (2004) 30 [hep-th/0408155].

[83] S. Deser and A. Waldron, *Arbitrary spin representations in de Sitter from dS/CFT with applications to dS supergravity*, *Nucl. Phys.* **B 662** (2003) 379 [hep-th/0301068].

[84] E.D. Skvortsov and M.A. Vasiliev, *Geometric formulation for partially massless fields*, *Nucl. Phys.* **B 756** (2006) 117 [hep-th/0601095].

[85] K.B. Alkalaev, *Two-column higher spin massless fields in AdS_d*, *Theor. Math. Phys.* **140** (2004) 1253 [hep-th/0311212].

[86] K.B. Alkalaev, O.V. Shaynkman and M.A. Vasiliev, *Lagrangian formulation for free mixed-symmetry bosonic gauge fields in $(A)dS_d$*, *JHEP* **08** (2005) 069 [hep-th/0501108].